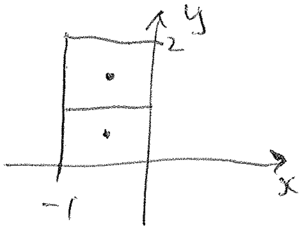
Math 223 **Exam 3** **Name:** Solution.

1. Consider the double integral $\displaystyle\iint_R f(x,y)\,dA$ where $f(x,y) = xy$ and R is the rectangle $-1 \le x \le 0$, $0 \le y \le 2$.

(a) (10 pts.) Estimate the double integral by a double Riemann sum where the partition consists of two squares with $\Delta x = 1$ and $\Delta y = 1$, and (x_i, y_j) are the centers of the squares.



$$f(-0.5, 0.5)\,\Delta x\,\Delta y + f(-0.5, 1.5)\,\Delta x\,\Delta y$$

$$= (-0.5)(0.5)\cdot 1 + (-0.5)(1.5)\cdot 1$$

$$= -0.25 + -0.75$$

$$= \underline{\underline{-1}}.$$

(b) (5 pts.) Calculate the exact value for the double integral.

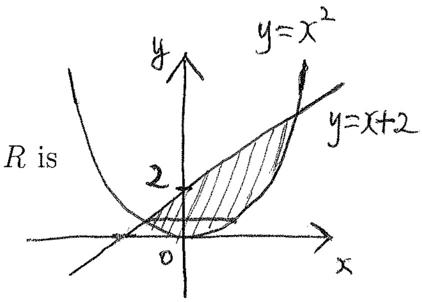
$$\int_0^2 \int_{-1}^0 xy\,dx\,dy$$

$$= \int_0^2 \frac{x^2}{2}y\Big|_{-1}^0\,dy = \int_0^2 0 - \frac{1}{2}y\,dy$$

$$= -\frac{1}{4}y^2\Big|_0^2$$

$$= \underline{\underline{-1}}.$$

2. Consider the double integral $\displaystyle\iint_R f(x,y)\,dA$ where $f(x,y) = x$ and R is the region above $y = x^2$ and below $y = x + 2$.



Find intersection.

$x^2 = x+2$

$x^2 - x - 2 = 0$

$(x-2)(x+1) = 0$

$x = 2$ or $x = -1$.

(a) (10 pts.) Write down the double integral if $dA = dy\,dx$.

$$\int_{-1}^{2}\int_{y=x^2}^{y=x+2} x \; dy\,dx$$

(b) (10 pts.) Write down the double integral if $dA = dx\,dy$.

$$\int_0^1 \int_{x=-\sqrt{y}}^{+\sqrt{y}} x\;dx\,dy + \int_1^4 \int_{x=y-2}^{\sqrt{y}} x\;dx\,dy.$$

(c) (10 pts.) Use part (a) or (b) to evaluate the double integral.

From (a), $\displaystyle\int_{-1}^{2}\int_{x^2}^{x+2} x\;dy\,dx$

$$= \int_{-1}^{2} xy\Big|_{y=x^2}^{x+2} dx = \int_{-1}^{2} x(x+2) - x^3\,dx$$

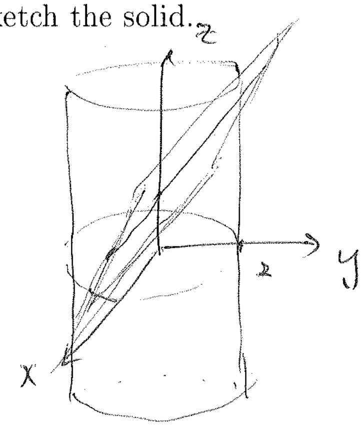
$$= \int_{-1}^{2} x^2 + 2x - x^3\,dx$$

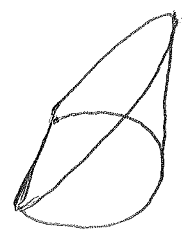
$$= \frac{x^3}{3} + x^2 - \frac{x^4}{4}\Big|_{-1}^{2}$$

$$= \left(\frac{8}{3} + 4 - 4\right) - \left(-\frac{1}{3} + 1 - \frac{1}{4}\right) = \underline{\underline{2\tfrac{1}{4}}}$$

3. A solid lies inside the cylinder $x^2 + y^2 = 4$, below the plane $z = y + 1$ and above the plane $z = 0$.

(a) (5 pts.) Sketch the solid.

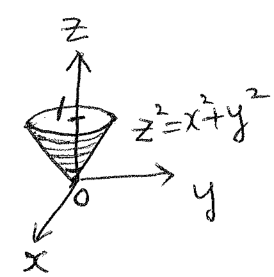




(b) (10 pts.) Suppose the density of the solid is given by $\delta(x, y, z) = x^2 + 1$. Set up a triple integral for its mass in rectangular coordinates.

$$\int_{-1}^{2} \int_{-\sqrt{4-y^2}}^{+\sqrt{4-y^2}} \int_{0}^{y+1} x^2+1 \; dz \, dx \, dy$$

4. (10 pts.) Set up a double integral for the surface area of the part of the cone $z^2 = x^2 + y^2$ that is above $z = 0$ and below $z = 1$.



$$z = f(x,y) = \sqrt{x^2+y^2}.$$

$$\frac{\partial f}{\partial x} = \frac{1}{2\sqrt{x^2+y^2}} \cdot 2x = \frac{x}{\sqrt{x^2+y^2}}$$

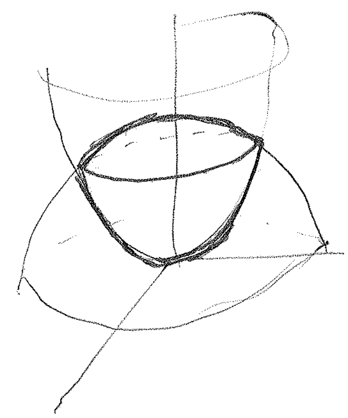
$$\frac{\partial f}{\partial y} = \frac{1}{2\sqrt{x^2+y^2}} \cdot 2y = \frac{y}{\sqrt{x^2+y^2}}$$

$$\text{Surface area} = \iint_R \sqrt{1 + \frac{x^2}{x^2+y^2} + \frac{y^2}{x^2+y^2}} \, dA$$

$$= \int_{-1}^{1} \int_{-\sqrt{1-x^2}}^{+\sqrt{1-x^2}} \sqrt{2} \, dy \, dx \quad \text{or} \quad \int_0^{2\pi} \int_0^1 \sqrt{2}\, r \, dr \, d\theta.$$

5. A solid is bounded above by the sphere $x^2 + y^2 + z^2 = 2$ and below by the paraboloid $z = x^2 + y^2$.

(a) (10 pts.) Sketch the solid and find the height, z, at which the two surfaces intersect.



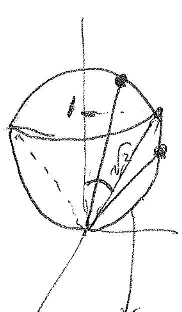
$$x^2+y^2+z^2=2$$
$$z=x^2+y^2$$
$$\Rightarrow z+z^2=2$$
$$z^2+z-2=0$$
$$(z+2)(z-1)=0$$
$$z=-2 \text{ or } \boxed{z=1}$$
$$\Downarrow$$
$$x^2+y^2=1$$

(b) (10 pts.) Set up a triple integral for its volume in cylindrical coordinates.

$$\int_0^{2\pi}\int_0^{1}\int_{r^2}^{\sqrt{2-r^2}} 1\cdot r\,dz\,dr\,d\theta$$

(c) (10 pts.) Set up a triple integral for its volume in spherical coordinates. (Hint: there should be two parts.)

$$\cos\phi=\frac{1}{\sqrt{2}} \Rightarrow \phi=\frac{\pi}{4}.$$



$$\int_0^{2\pi}\int_0^{\frac{\pi}{4}}\int_0^{\sqrt{2}} \rho^2\sin\phi\,d\rho\,d\phi\,d\theta$$
$$+\int_0^{2\pi}\int_{\frac{\pi}{4}}^{\frac{\pi}{2}}\int_0^{\frac{\cos\phi}{\sin^2\phi}} \rho^2\sin\phi\,d\rho\,d\phi\,d\theta.$$

$$z=x^2+y^2=r^2$$
$$\rho\cos\phi=(\rho\sin\phi)^2$$
$$\rho\cos\phi=\rho^2\sin^2\phi$$
$$\rho=\frac{\cos\phi}{\sin^2\phi}$$