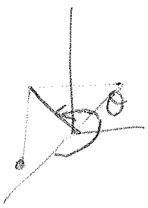
Math 223 **Exam 2** **Name:** Solution

1. (10 pts.) Suppose that the rectangular coordinates of P is $(-3, -4, -5)$. Find the cylindrical coordinates and spherical coordinates of P (compute θ and ϕ out).

Cylindrical



$$r = \sqrt{x^2+y^2} = \sqrt{(-3)^2+(-4)^2} = \underline{5}$$

$$\theta = \tan^{-1}\frac{y}{x} = \tan^{-1}\frac{-4}{-3} = \underline{0.927+\pi}$$

$$z = \underline{-5}$$

Spherical

$$\rho = \sqrt{x^2+y^2+z^2} = \underline{\sqrt{50}}$$

$$\theta = \underline{0.927+\pi}$$

$$\phi = \cos^{-1}\frac{z}{\rho} = \cos^{-1}\frac{-5}{\sqrt{50}} = \underline{\frac{3\pi}{4}}$$

2. Let $f(x, y) = x\sqrt{y-x}$.
(a) (10 pts.) Find $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$.

$$\frac{\partial f}{\partial x} = 1\cdot\sqrt{y-x} + x\cdot\frac{1}{2\sqrt{y-x}}\cdot(-1) \quad //$$

$$\frac{\partial f}{\partial y} = x\cdot\frac{1}{2\sqrt{y-x}} \quad //$$

(b) (5 pts.) What is the largest domain of $\frac{\partial f}{\partial x}$?

$$y - x > 0 \qquad \boxed{y > x}$$

(c) (5 pts.) Suppose $x = s^2 + t^2$ and $y = e^{st}$. Find $\frac{\partial f}{\partial t}$ and express your answer in terms of s and t only.

By Chain rule, $\frac{\partial f}{\partial t} = \frac{\partial f}{\partial x}\cdot\frac{\partial x}{\partial t} + \frac{\partial f}{\partial y}\cdot\frac{\partial y}{\partial t}$

$$\frac{\partial f}{\partial t} = \left(\sqrt{y-x} - \frac{x}{2\sqrt{y-x}}\right)\cdot 2t + \frac{x}{2\sqrt{y-x}}\cdot e^{st}\cdot s$$

$$= \underline{\left(\sqrt{e^{st}-s^2-t^2} - \frac{s^2+t^2}{2\sqrt{e^{st}-s^2-t^2}}\right)\cdot 2t + \frac{s^2+t^2}{2\sqrt{e^{st}-s^2-t^2}}\cdot e^{st}\cdot s}$$

3. Assume that $z = f(x, y)$ satisfies $\sin(xz) + yz = 1$.
(a) (3 pts.) What is z when $x = 0$ and $y = 1$?

$$\sin(0 \cdot z) + 1 \cdot z = 1$$
$$0 + z = 1 \quad \Rightarrow \quad \boxed{z = 1}$$

(b) (12 pts.) Find $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$ at $x = 0$ and $y = 1$.

$$\frac{\partial \sin(xz)}{\partial x} + y\frac{\partial z}{\partial x} = 0$$
$$\cos(xz) \cdot \frac{\partial xz}{\partial x} + y\frac{\partial z}{\partial x} = 0$$
$$\cos(xz)\left(z + x\frac{\partial z}{\partial x}\right) + y\frac{\partial z}{\partial x} = 0$$
$$x\cos(xz)\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial x} = -z\cos(xz)$$

$$\boxed{\therefore \frac{\partial z}{\partial x} = \frac{-z\cos(xz)}{x\cos(xz) + y}.}$$

$$\frac{\partial \sin(xz)}{\partial y} + 1 \cdot z + y\frac{\partial z}{\partial y} = 0$$
$$\cos(xz) \cdot x \cdot \frac{\partial z}{\partial y} + z + y\frac{\partial z}{\partial y} = 0$$

$$\boxed{\therefore \frac{\partial z}{\partial y} = \frac{-z}{x\cos(xz) + y}}$$

At $x = 0,\ y = 1,\ z = 1$

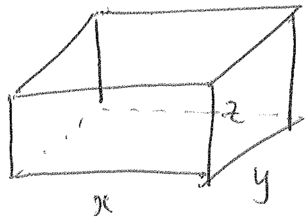
$$\frac{\partial z}{\partial x} = \underline{\underline{-1}}, \qquad \frac{\partial z}{\partial y} = \underline{\underline{-1}}$$

(c) (5 pts.) Use (b) to estimate the value of z when $x = 0.02$ and $y = 0.99$.

$$\Delta f \approx \frac{\partial f}{\partial x} \cdot \Delta x + \frac{\partial f}{\partial y} \cdot \Delta y$$
$$= (-1)(0.02) + (-1)(-0.01) = -0.01$$

$$\therefore f(0.02, 0.99) - f(0, 1) \approx -0.01$$
$$f(0.02, 0.99) \approx 1 - 0.01 = \underline{0.99}.$$

4. (15 pts.) We want to build an open-topped box with a volume of 500 inch^3. The material for its bottom and four sides costs \$ 0.5/inch^2. Find the dimensions of the box that minimizes the total cost.



$$xyz = 500$$
$$z = \frac{500}{xy}$$

$$\text{Cost} = 0.5xy + 0.5 \cdot 2xz + 0.5 \cdot 2yz$$
$$C = 0.5xy + xz + yz$$
$$C = 0.5xy + \frac{500}{y} + \frac{500}{x}.$$

$$\frac{\partial C}{\partial x} = 0.5y - \frac{500}{x^2} = 0 \implies y = \frac{1000}{x^2}$$
$$\frac{\partial C}{\partial y} = 0.5x - \frac{500}{y^2} = 0 \implies x = \frac{1000}{y^2}.$$

So, $$x = \frac{1000}{\left(\frac{1000}{x^2}\right)^2} = \frac{\frac{1000}{1}}{\frac{1000^2}{x^4}} = \frac{x^4}{1000}$$

$$x^4 - 1000x = 0$$
$$x(x^3 - 1000) = 0$$
$$x = 0 \quad \text{or} \quad x = 10$$
(not possible)

$$y = \frac{1000}{10^2} = 10$$

$$z = \frac{500}{xy} = \frac{500}{10 \cdot 10} = 5$$

$\therefore$ The dimensions are $\boxed{10 \times 10 \times 5}$

5. $P(x, y, z) = 100 + xy - yz$ denotes the pressure in a room at (x, y, z).
(a) (5 pts.) Find $\vec{\nabla} P(x, y, z)$.

$$\vec{\nabla} P = \left\langle \frac{\partial P}{\partial x}, \frac{\partial P}{\partial y}, \frac{\partial P}{\partial z} \right\rangle$$

$$= \langle y, x-z, -y \rangle \;//$$

(b) (5 pts.) Let $\vec{v} = \langle 1, 1, 1 \rangle$. Find the directional derivative of P at $(1, 2, 3)$ in the direction of $\vec{v}$.

$$\vec{u} = \frac{\vec{v}}{|\vec{v}|} = \frac{\langle 1, 1, 1 \rangle}{\sqrt{3}}$$

$$D_{\vec{u}} P(1,2,3) = \vec{\nabla} P(1,2,3) \cdot \vec{u}$$

$$= \langle 2, -2, -2 \rangle \cdot \frac{\langle 1, 1, 1 \rangle}{\sqrt{3}}$$

$$= \boxed{-\frac{2}{\sqrt{3}}}$$

(c) (5 pts.) Find a possible direction $\vec{u}$ such that $D_{\vec{u}} P(1, 2, 3) = 0$.

We want $\vec{\nabla} P(1,2,3) \cdot \vec{u} = 0$, say $\vec{u} = \langle x, y, z \rangle$.

$$\langle 2, -2, -2 \rangle \cdot \langle x, y, z \rangle = 0$$

$$2x - 2y - 2z = 0$$

There are infinitely many solutions. For example $x=1,\ y=1,\ z=0$

$\langle 1, 1, 0 \rangle \rightsquigarrow$ unit vector $\vec{u} = \underline{\left\langle \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0 \right\rangle}$

(c) (5 pts.) Suppose that a dust particle moves in the direction where the rate of change of pressure is <u>most negative.</u> In what direction does a dust particle move when it is at $(1, 2, 3)$?

$$D_{\vec{u}} P(1,2,3) = \vec{\nabla} P(1,2,3) \cdot \vec{u}$$

$$= |\vec{\nabla} P(1,2,3)| \cdot |\vec{u}| \cos\theta$$

which is most negative when $\cos\theta = -1$ or $\boxed{\theta = \pi}$.

This means that $\vec{u}$ is in the opposite direction of $\vec{\nabla} P(1,2,3)$

$$\vec{u} = \frac{-\langle 2, -2, -2 \rangle}{\sqrt{2^2 + 2^2 + 2^2}} = \left\langle \frac{-2}{\sqrt{12}}, \frac{2}{\sqrt{12}}, \frac{2}{\sqrt{12}} \right\rangle = \underline{\left\langle \frac{-1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}} \right\rangle}$$

6. True or False (3 pts. each)

(a) $u = e^t \sin x$ is a solution to $\frac{\partial^2 u}{\partial t^2} + \frac{\partial^2 u}{\partial x^2} = 0$.

True / False

(b) $x^2 - y^2 - z^2 = -1$ is a hyperboloid of 2 sheets.

True / False $\quad -x^2+y^2+z^2=1$.

(c) There is a function $f(x,y)$ satisfying $f_x = x^2$ and $f_y = y^2$.

True / False

$f_x = x^2 \longrightarrow f = \frac{1}{3}x^3 + g(y)$

$f_y = y^2 \longrightarrow f = \frac{1}{3}y^3 + h(x)$

$\Rightarrow f = \frac{1}{3}x^3 + \frac{1}{3}y^3 + C$.

(d) The level curves of $z = \sin(x+y)$ are parallel straight lines.

True / False

$z=0$, $\sin(x+y)=0$, $x+y = 0, \pi, 2\pi, \ldots$

$z=1$, $\sin(x+y)=1$, $x+y = \frac{\pi}{2}, \frac{\pi}{2}+2\pi, \ldots$ ← parallel straight lines.

(e) The surface given by the spherical equation, $\phi = 1$, is a plane.

True / False $\quad \phi=1$ is a cone.

Formulas:

$r = \sqrt{x^2+y^2}$, $\theta = \tan^{-1}\frac{y}{x}$.

$\rho = \sqrt{x^2+y^2+z^2}$, $\phi = \cos^{-1}\frac{z}{\sqrt{x^2+y^2+z^2}} = \tan^{-1}\frac{\sqrt{x^2+y^2}}{z}$.

$x = r\cos\theta$, $y = r\sin\theta$.

$x = \rho\sin\phi\cos\theta$, $y = \rho\sin\phi\sin\theta$, $z = \rho\cos\phi$.

$\Delta f \approx \vec{\nabla} f \cdot \langle \Delta x, \Delta y\rangle = \frac{\partial f}{\partial x}\Delta x + \frac{\partial f}{\partial y}\Delta y$. $\qquad D_{\vec{u}} f = \vec{\nabla} f \cdot \vec{u} = |\vec{\nabla} f||\vec{u}|\cos\theta$.