Math 223 **Exam 1** **Name:** Solution

1. Let $\vec{u} = \langle 2, -1, 3 \rangle$ and $\vec{v} = \langle 4, x, -2 \rangle$.

(a) (5 pts.) Find x so that $\vec{u}$ and $\vec{v}$ are perpendicular.

$$\vec{u} \cdot \vec{v} = 0$$

$$8 - x - 6 = 0$$

$$\boxed{x = 2}$$

(b) (10 pts.) Is there a value for x so that $\vec{u}$ and $\vec{v}$ are parallel? Explain your answer.

$$\vec{u} = k\vec{v}$$

$$\langle 2, -1, 3 \rangle = k \langle 4, x, -2 \rangle = \langle 4k, kx, -2k \rangle$$

$$\begin{cases} 2 = 4k \longrightarrow & k = \frac{1}{2} \\ -1 = kx & \\ 3 = -2k \longrightarrow & k = \frac{3}{-2} \end{cases}$$

So, there is no single solution of k that satisfies $\vec{u} = k\vec{v}$. Therefore, $\vec{u}$ and $\vec{v}$ can never be parallel no matter what x is.

2. Given the two lines $l_1 : \begin{cases} x = 1 - 3t \\ y = 1 + t \\ z = 1 + t \end{cases}$ and $l_2 : \begin{cases} x = 2 + t \\ y = 3 + 2t \\ z = -1 - 2t \end{cases}$.

(a) (15 pts.) Show that l_1 and l_2 intersect and find the point of intersection.

$$\begin{cases} x = 1-3t \\ y = 1+t \\ z = 1+t \end{cases} \qquad \begin{cases} x = 2+s \\ y = 3+2s \\ z = -1-2s \end{cases}$$

$$\begin{cases} 1-3t = 2+s & \text{——— (1)} \\ 1+t = 3+2s & \text{——— (2)} \\ 1+t = -1-2s & \text{——— (3)} \end{cases}$$

(2) − (3)

$$0 = 4+4s \Rightarrow s = -1$$

Put $s=-1$ into (3),

$$1+t = -1+2 \Rightarrow t = 0$$

Put $s=-1$ and $t=0$ into (1) to check

$$1 = 1-3\cdot 0 \stackrel{?}{=} 2+(-1) = 1 \quad \checkmark$$

So, l_1 and l_2 intersect.

The point of intersection is

$$\begin{aligned} x &= 1-3\cdot 0 = 1 \\ y &= 1+0 = 1 \\ z &= 1+0 = 1 \end{aligned}$$

$$\boxed{(1, 1, 1)}.$$

Note: One may want to check that $\langle -3, 1, 1 \rangle$ and $\langle 1, 2, -2 \rangle$ are not parallel. But this is not necessary since we show that the two line intersects at a point above.

(b) (10 pts.) Find the angle between l_1 and l_2.

$$\vec{V_1} = \langle -3, 1, 1 \rangle, \quad \vec{V_2} = \langle 1, 2, -2 \rangle$$

$$\cos\theta = \frac{\vec{V_1} \cdot \vec{V_2}}{|\vec{V_1}|\,|\vec{V_2}|} = \frac{-3+2-2}{\sqrt{(-3)^2+1^2+1^2}\,\sqrt{1^2+2^2+(-2)^2}} = \frac{-3}{\sqrt{11}\cdot\sqrt{9}}$$

$$= \frac{-1}{\sqrt{11}}$$

$$\theta = \cos\left(\frac{-1}{\sqrt{11}}\right).$$

Note: Actually there are two correct answers.

(c) (10 pts.) Find the equation of the plane containing l_1 and l_2.

$$\vec{n} = \vec{V_1} \times \vec{V_2} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ -3 & 1 & 1 \\ 1 & 2 & -2 \end{vmatrix} = \vec{i}\begin{vmatrix} 1 & 1 \\ 2 & -2 \end{vmatrix} - \vec{j}\begin{vmatrix} -3 & 1 \\ 1 & -2 \end{vmatrix} + \vec{k}\begin{vmatrix} -3 & 1 \\ 1 & 2 \end{vmatrix}$$

$$= \langle -4, -5, -7 \rangle$$

$\therefore \quad -4x - 5y - 7z = d$

$(1, 1, 1)$ is on the plane

$$-4 - 5 - 7 = d$$

$$d = -16.$$

$\therefore$ $\boxed{\begin{array}{l} -4x - 5y - 7z = -16 \\ \text{or } 4x + 5y + 7z = 16 \end{array}}$

3. Let $\vec{r}(t) = \langle \frac{2}{3}t^3, t^2, t \rangle$ be a curve in the space.

(a) (10 pts.) Find $\vec{r'}(t)$ and $\vec{r''}(t)$.

$$\vec{r}'(t) = \langle 2t^2, 2t, 1 \rangle.$$

$$\vec{r}''(t) = \langle 4t, 2, 0 \rangle.$$

(b) (15 pts.) Find the parametric equations of the tangent line to $\vec{r}(t)$ at $t = 1$.

We need a point and a direction/velocity.

Point: $\vec{r}(1) = \langle \frac{2}{3}, 1, 1 \rangle$

Direction: $\vec{r}'(1) = \langle 2, 2, 1 \rangle$

$$\boxed{\begin{cases} x = \frac{2}{3} + 2t \\ y = 1 + 2t \\ z = 1 + t \end{cases}}$$

(c) (10 pts.) Find the arc length of $\vec{r}(t)$ from $t = 1$ to $t = 2$.

$$S = \int_1^2 \underset{\substack{\uparrow \\ \text{speed}}}{v(t)}\, dt = \int_1^2 \sqrt{(2t^2)^2 + (2t)^2 + 1^2}\, dt$$

$$= \int_1^2 \sqrt{4t^4 + 4t^2 + 1}\, dt = \int_1^2 \sqrt{(2t^2+1)^2}\, dt$$

$$= \int_1^2 2t^2 + 1\, dt = \frac{2}{3}t^3 + t \Big|_1^2 = \boxed{\frac{17}{3}}$$

(d) (10 pts.) Find the curvature of $\vec{r}(t)$ at $t = 0$.

$$K = \frac{|\vec{r}\,' \times \vec{r}\,''|}{|\vec{r}\,'|^3}$$

$$\vec{r}\,'(0) = \langle 0, 0, 1 \rangle$$
$$\vec{r}\,''(0) = \langle 0, 2, 0 \rangle$$

$$\vec{r}\,'(0) \times \vec{r}\,''(0) = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 0 & 0 & 1 \\ 0 & 2 & 0 \end{vmatrix} = -2\vec{i} + 0\vec{j} + 0\vec{k}$$

$$\therefore K = \frac{\sqrt{(-2)^2 + 0^2 + 0^2}}{\left(\sqrt{0^2 + 0^2 + 1^2}\right)^3} = \frac{2}{1} = \boxed{2}$$

(e) (5 pts.) What is the radius of a circle that fits closely to $\vec{r}(t)$ near $t = 0$?

$$\rho = \frac{1}{K} = \boxed{\frac{1}{2}}$$