1 Compute the following improper integrals. Show all work necessary to justify your answer.

(a) $\int_{-\infty}^{0} xe^{x^2}\,dx$

Solution: This is an improper integral, which means that there is a limit involved when evaluating the endpoints. We will start with the indefinite integral (or anti-derivative), using the substitution $u = x^2$, $du = 2x\,dx$, and $x\,dx = \frac{1}{2}\,du$:

$$\begin{aligned}\int xe^{x^2}\,dx &= \int e^{x^2} x\,dx = \int e^u \frac{1}{2}\,du\\ &= \frac{1}{2}\int e^u\,du = \frac{1}{2}e^u + K\\ &= \frac{1}{2}e^{x^2} + K.\end{aligned}$$

Since we're computing a definite integral, we need only take the simplest of these anti-derivatives: $\frac{1}{2}e^{x^2}$. Thus

$$\begin{aligned}\int_{-\infty}^{0} xe^{x^2}\,dx &= \lim_{a\to-\infty}\int_a^0 xe^{x^2}\,dx\\ &= \lim_{a\to-\infty}\left[\frac{1}{2}e^{x^2}\right]_a^0\\ &= \lim_{a\to-\infty}\left[\frac{1}{2}e^0 - \frac{1}{2}e^{a^2}\right]\\ &= \frac{1}{2} - \frac{1}{2}\lim_{a\to-\infty} e^{a^2}\\ &= -\infty.\end{aligned}$$

Thus, the integral we are trying to compute does not exist. That is, the integral is divergent.

(b) $\int_1^{\infty} \frac{3x}{(x+1)^3}\,dx$

Solution: Here a simple way to evaluate this integral is via the substitution $u = x + 1$. Thus $x = u - 1$, $dx = du$, and we can change the limits to $u = 2$ when $x = 1$ and $u \to \infty$ when $x \to \infty$. Hence the definite integral becomes

$$\begin{aligned}\int_1^{\infty} \frac{3x}{(x+1)^3}\,dx &= \int_2^{\infty} \frac{3(u-1)}{u^3}\,du\\ &= 3\int_2^{\infty}\left(\frac{u}{u^3} - \frac{1}{u^3}\right)du\\ &= 3\int_2^{\infty}\left(u^{-2} - u^{-3}\right)du\\ &= 3\lim_{b\to\infty}\int_2^{b}\left(u^{-2} - u^{-3}\right)du\\ &= 3\lim_{b\to\infty}\left[-u^{-1} + \frac{1}{2}u^{-2}\right]_2^b\\ &= 3\lim_{b\to\infty}\left[\left(-\frac{1}{b} + \frac{1}{2b^2}\right) - \left(-\frac{1}{2} + \frac{1}{2}\cdot\frac{1}{2^2}\right)\right]\\ &= 3\left[(-0+0) - \left(-\frac{1}{2} + \frac{1}{2}\cdot\frac{1}{2^2}\right)\right]\\ &= 3\left(\frac{1}{2} - \frac{1}{8}\right) = \frac{9}{8}.\end{aligned}$$

(c) $\displaystyle\int_0^1 \frac{1}{\sqrt{x}}\,dx$

Solution: This is an improper integral because the function $\frac{1}{\sqrt{x}} = x^{-1/2}$ is undefined at $x = 0$. Hence we *must* use a limit to compute this integral:

$$\begin{aligned}\int_0^1 x^{-1/2}\,dx &= \lim_{a\to 0^+}\int_a^1 x^{-1/2}\,dx\\ &= \lim_{a\to 0^+}\left(\left.\frac{x^{1/2}}{1/2}\right|_a^1\right)\\ &= \lim_{a\to 0^+}\left(2\sqrt{x}\Big|_a^1\right)\\ &= \lim_{a\to 0^+}\left(2\sqrt{1}-2\sqrt{a}\right)\\ &= (2-0) = 2.\end{aligned}$$

Thus this integral evaluates to 2.

2 Consider a random variable X whose probabilities are expressed using a density $f(x) = k\sqrt{x+10}$, $-6 \le x \le 6$.

(a) For what value of k is the function $f(x)$ a valid probability density function?

Solution: The value of k required is the one that makes $\displaystyle\int_a^b f(x)\,dx = 1$. In this case, this means

$$1 = \int_{-6}^{6} k\sqrt{x+10}\,dx.$$

Making the substitution $u = x + 10$, we compute the anti-derivative:

$$\int k\sqrt{x+10}\,dx = \int k\sqrt{u}\,du = k\int u^{1/2}\,du = \frac{2k}{3}u^{3/2} + K. = \frac{2k}{3}(x+10)^{3/2} + K.$$

Since this is a definite integral, we don't need the arbitrary constant, we simply need to evaluate the limits:

$$\begin{aligned}1 &= \int_{-6}^{6} k\sqrt{x+10}\,dx\\ &= \left[\frac{2k}{3}(x+10)^{3/2}\right]_{-6}^{6}\\ &= \frac{2k}{3}\left[(6+10)^{3/2} - (-6+10)^{3/2}\right]\\ &= \frac{2k}{3}(64-8)\\ &= \frac{112k}{3}.\end{aligned}$$

Thus, solving for k, we get $k = 3/112 \approx 0.0267857$.

(b) Compute $\Pr(X \leq 0)$.

Solution: We compute:

$$\begin{aligned}
\Pr(X \leq 0) &= \int_{-6}^{0} k\sqrt{x+10}\,dx \\
&= \left[\frac{2k}{3}(x+10)^{3/2}\right]_{-6}^{0} \\
&= \frac{2k}{3}\left((0+10)^{3/2} - (-6+10)^{3/2}\right) \\
&= \frac{1}{56}\left(10\sqrt{10} - 8\right) \\
&\approx 0.4218353.
\end{aligned}$$

(Here we've used the fact that $k = 3/112$ to simplify $2k/3$ to $1/56$.)

(c) Find the expected value (or mean) $\mu = E(X)$.

Solution: The expected value is

$$\mu = E(X) = \int_{-6}^{6} x \cdot k\sqrt{x+10}\,dx.$$

It is simplest to make the substitution $u = x + 10$, so $x = u - 10$ and $dx = du$. Also, $u = 16$ when $x = 6$ and $u = 4$ when $x = -6$. Thus

$$\begin{aligned}
\mu = E(X) &= \int_{4}^{16} (u-10)\cdot\frac{3}{112}\sqrt{u}\,du \\
&= \frac{3}{112}\int_{4}^{16}\left(u^{3/2} - 10u^{1/2}\right)\,du \\
&= \frac{3}{112}\int_{4}^{16}\left(u^{3/2} - 10u^{1/2}\right)\,du \\
&= \frac{3}{112}\left(\frac{u^{5/2}}{5/2} - 10\frac{u^{3/2}}{3/2}\right)\Bigg|_{4}^{16} \\
&= \frac{3}{112}\left(\frac{2}{5}u^{5/2} - \frac{20}{3}u^{3/2}\right)\Bigg|_{4}^{16} \\
&= \frac{3}{112}\left[\left(\frac{2}{5}(16)^{5/2} - \frac{20}{3}(16)^{3/2}\right) - \left(\frac{2}{5}(4)^{5/2} - \frac{20}{3}(4)^{3/2}\right)\right] \\
&= \frac{3}{112}\left[\left(\frac{2}{5}(1024) - \frac{20}{3}(64)\right) - \left(\frac{2}{5}(32) - \frac{20}{3}(8)\right)\right] \\
&= \frac{3}{112}\left(\frac{2048}{5} - \frac{1280}{3} - \frac{64}{5} + \frac{160}{3}\right) \\
&= \frac{3}{112}\cdot\frac{352}{15} = \frac{22}{35} \approx 0.62857.
\end{aligned}$$

This is our expected value.

3 Suppose an urn contains fourteen balls: five red and nine green. A ball is pulled from the urn, its color is noted, and then it is returned to the urn. This is repeated four times.

(a) What is the probability that three of the four balls drawn from the urn are green?

Solution: This is a Bernoulli Trial – we're asked for the number of successes (green balls) in four independent trials. On each trial, the probability of a successful green ball is 9 out of 14, or $p = \frac{9}{14}$. Thus the probability that we exactly three of the four balls drawn will be green is

$$b(4, 3; {}^9\!/_{14}) = \binom{4}{3}\left(\frac{9}{14}\right)^3\left(\frac{5}{14}\right) \approx 0.3795.$$

This is the probability we want to find.

(b) Let X be the random variable that represents the number of green balls drawn. List the values of X together with the probability distribution.

Solution: The random variable X can take on integer values from 0 to 4. The probabilities are calculated as in part (a), and are shown below:

k	0	1	2	3	4
$\Pr(X = k)$	$b(4, 0; {}^9\!/_{14})$ $= \binom{4}{0}\left(\frac{5}{14}\right)^4$ ≈ 0.0163	$b(4, 1; {}^9\!/_{14})$ $= \binom{4}{1}\left(\frac{9}{14}\right)^1\left(\frac{5}{14}\right)^3$ ≈ 0.1171	$b(4, 2; {}^9\!/_{14})$ $= \binom{4}{2}\left(\frac{9}{14}\right)^2\left(\frac{5}{14}\right)^2$ ≈ 0.3163	$b(4, 3; {}^9\!/_{14})$ $= \binom{4}{3}\left(\frac{9}{14}\right)^3\left(\frac{5}{14}\right)^1$ ≈ 0.3795	$b(4, 4; {}^9\!/_{14})$ $= \binom{4}{4}\left(\frac{9}{14}\right)^4$ ≈ 0.1708

A good check is that these probabilities sum to 1:

$$0.0163 + 0.1171 + 0.3163 + 0.3795 + 0.1708 = 1.0000.$$

(c) Find $E(X)$, the expected value of this random variable X.

Solution: We use the probabilities in the table above to compute:

$$E(X) \approx 0(0.0163) + 1(0.1171) + 2(0.3163) + 3(0.3795) + 4(0.1708) \approx 2.571.$$

(We've rounded in computing these probabilities, so we'll keep even fewer decimals in our expected value.)

Another, simpler approach is to use the fact that the expected value of a Bernoulli Trial is $E(X) = n \cdot p$, where $n = 4$ is the number of trials and $p = 9/14$ is the probability of a green success. Thus

$$E(X) = n \cdot p = 4 \cdot \frac{9}{14} = \frac{18}{7} \approx 2.571,$$

as above.

4 The starting salary of graduates from Standard American University is a normally distributed random variable with a mean of \$30,000 per year and a standard deviation of \$10,000 per year. (You may use a table of Z-values.)

(a) What is the likelihood that a new graduate will start with a salary of more than \$50,000 per year.

Solution: Recall that, if X is a normal distribution with mean $E(X) = \mu$ and standard deviation $\sqrt{\text{Var}(X)} = \sigma$, then

$$\Pr(a \le X \le b) = \Pr\left(\frac{a - \mu}{\sigma} \le Z \le \frac{b - \mu}{\sigma}\right),$$

where Z is the standard normal distribution. Here X is a normal distribution with mean $\mu = \$30{,}000$ and a standard deviation $\sigma = \$10{,}000$. Hence

$$\begin{aligned}\Pr(X > \$50,000) &= \Pr\left(Z > \frac{\$50,000 - \$30,000}{\$10,000}\right)\\ &= \Pr(Z > 2).\end{aligned}$$

We have no tables that show values for $\Pr(Z > 2)$; instead, our tables show $\Pr(0 \le Z \le 2)$ (this Z-value is 0.4772). But $\Pr(Z \ge 0) = 1/2$ (since 0 is the mean of our symmetric standard normal distribution); thus

$$\frac{1}{2} = \Pr(0 \le Z) = \Pr(0 \le Z \le 2) + \Pr(2 < Z) = 0.4772 + \Pr(Z > 2).$$

Thus $\Pr(Z > 2) = 0.5 - 0.4772 = 0.0228$. Hence the probability that a new graduate will start with a salary of more than \$50,000 per year is roughly 2.28%.

(b) Jessica has been given the choice of going to Standard U or enrolling in a training program at Jobs R Us. Clients of Jobs R Us hire graduates of the program for various positions. Each year 15% are hired as managers, 50% as clerks, 20% as assistants and 15% as technicians. Starting salaries are set at \$42,000 per year for managers, \$28,000 per year for clerks, \$25,000 per year for assistants and \$22,000 per year for technicians.

On the basis of starting salary alone, where should Jessica apply?

Solution: We will make Jessica's decision for her based on her expected starting salary. At Standard U., her expected starting salary would be $E(X) = \mu = \$30{,}000$ (as in part (a)). At Jobs R Us, on the other hand, her starting salary would be based on the following random variable (which we'll call Y, as we've already used X):

Starting Salary:	\$42,000	\$28,000	\$25,000	\$22,000
Probability:	15%	50%	20%	15%

Then the expected value of this random variable Y is

$$E(Y) = \$42,000(0.15) + \$28,000(.50) + \$25,000(.20) + \$22,000(.15) = \$28,600.$$

Thus the Standard U. starting salary is slightly higher (on average), so Jessica should attend the university.

5 The amount of time required to serve a customer at a pizza restaurant during lunch time is an exponential random variable whose average is 5 minutes. The restaurant promises to give a full refund if the serving time for a customer is more than L minutes.

(a) What is the value of the constant λ that is used in this exponential density function?

Solution: Recall that an exponential random variable X has probability density function $f(x) = \lambda e^{-\lambda x}$ and mean $E(X) = \frac{1}{\lambda}$. Since them mean here is 5, we have $\frac{1}{\lambda} = 5$, or $\lambda = 1/5$.

(b) What is the probability that a customer is served in under 5 minutes?

Solution: This asks for

$$\Pr(0 \le X \le 5) = \int_0^5 \lambda e^{-\lambda x}\, dx.$$

We compute this integral with $\lambda = 1/5$:

$$\begin{aligned}\Pr(0 \le X \le 5) &= \int_0^5 \lambda e^{-\lambda x}\, dx\\ &= -e^{-\lambda x}\Big|_0^5\\ &= -e^{-5\lambda} - (-e^0)\\ &= 1 - e^{-1} \approx 0.6321205588.\end{aligned}$$

That is, the probability that a customer is served in under 5 minutes is roughly 63.21%.

(c) What is L so that the restaurant only needs to refund 0.5% of the customers?

Solution: Here we wish to find L so that

$$\Pr(0 \le X \le L) = \int_0^L \lambda e^{-\lambda x}\,dx = 0.995.$$

We compute this integral as in part (b):

$$\begin{aligned} 0.995 = \Pr(0 \le X \le L) &= \int_0^L \lambda e^{-\lambda x}\,dx \\ &= -e^{-\lambda x}\Big|_0^L \\ &= 1 - e^{-L/5}. \end{aligned}$$

Solving $1 - e^{-L/5} = 0.995$ for L, we get $L = -5\ln(0.005) \approx 26.491$. That is, if $L = 26.5$ minutes, only 0.5% of the restaurant's customers will require a refund.

6 In a carnival game, a die is rolled that has three blue sides, two red sides and a green side. Fifty cents is placed on the color of your choice and the die is rolled. A match on blue pays fifty cents, a match on red pays \$1 and a match on green pays \$2. Let X be a random variable representing the payoff of a bet on red.

(a) Make a probability table for X.

Solution: Here is a probability table for X:

Die Color:	Red	Blue	Green
Payoff of a bet on red:	\$1	\$0	\$0
Probability:	$\frac{2}{6} = \frac{1}{3}$	$\frac{3}{6} = \frac{1}{2}$	$\frac{1}{6}$

(b) What is the average result? That is, what is the expected payoff?

Solution: The expected payoff from a bet on red is

$$E(X) = \$1 \cdot \frac{1}{3} + \$0 \cdot \frac{1}{2} + \$0 \cdot \frac{1}{6} = \$\frac{1}{3}.$$

Thus the expected payoff from a bet on red is a third of a dollar.

7 Let $f(x) = \frac{1}{9}x^2$, $0 \le x \le 3$.

(a) Verify that $f(x)$ is a probability density function.

Solution: This problem asks us to check that

$$\int_a^b f(x)\,dx = 1 \qquad \text{and} \qquad f(x) \ge 0 \text{ for } x \text{ in the interval } [a,b],$$

as these are the conditions for $f(x)$ to be a probability density function on $[a,b]$. Clearly $f(x) \ge 0$, so we must only check that

$$\int_0^3 \frac{1}{9}x^2\,dx = 1.$$

This is straightforward:

$$\begin{aligned} \int_0^3 \frac{1}{9}x^2\,dx &= \frac{1}{9}\cdot\frac{1}{3}x^3\Big|_0^3 \\ &= \frac{1}{27}\cdot\left(3^3 - 0^3\right) \\ &= 1. \end{aligned}$$

(b) Find the expected value (or mean) $\mu = E(x)$.

Solution: We compute:

$$\begin{aligned}\mu = E(X) &= \int_a^b x \cdot f(x)\,dx = \int_0^3 x \cdot \frac{1}{9}x^2\,dx \\ &= \frac{1}{9}\int_0^3 x^3\,dx \\ &= \frac{1}{9}\cdot\frac{1}{4}x^4\Big|_0^3 \\ &= \frac{1}{9}\cdot\frac{1}{4}\left(3^4 - 0^4\right) \\ &= \frac{1}{9}\cdot\frac{1}{4}\cdot 81 \\ &= \frac{9}{4} = 2.25.\end{aligned}$$

This is $E(X)$.

(c) Compute $\Pr(1 \le X \le 2)$.

Solution: Recall that

$$\Pr(1 \le X \le 2) = \int_1^2 f(x)\,dx.$$

We have already integrated in part (a) to find that $F(x) = \frac{1}{9}\cdot\frac{1}{3}x^3 = \frac{1}{27}x^3$ is an anti-derivative of $f(x)$. Thus our probability simplifies to

$$\Pr(1 \le X \le 2) = \int_1^2 f(x)\,dx = F(x)\Big|_1^2 = F(2) - F(1).$$

Plugging in $F(x) = \frac{1}{27}x^3$, we get

$$\Pr(1 \le X \le 2) = F(2) - F(1) = \frac{1}{27}(2)^3 - \frac{1}{27}(1)^3 = \frac{7}{27}.$$

Thus $\Pr(1 \le X \le 2) = \dfrac{7}{27} \approx 0.2593$.

8 The number of customers coming into a store was recorded every hour for 12 hours. The following observations were made.

During 4 of the one-hour intervals, there was one customer.

During 6 of the one-hour intervals, there were two customers.

During 2 of the one-hour intervals, there were four customers.

A one-hour interval is to be selected at random and the number of customers noted. Let X be the outcome. Then X is a random variable taking on values between 1 and 4.

(a) Write out a probability table for X.

Solution: Here is a probability table for X:

Number of Customers:	1	2	3	4
Probability:	$\frac{4}{12} = \frac{1}{3}$	$\frac{6}{12} = \frac{1}{2}$	$\frac{0}{12} = 0$	$\frac{2}{12} = \frac{1}{6}$

(b) Compute the expected value $E(X)$.

Solution: The expected value is given by

$$E(X) = 1 \cdot \frac{1}{3} + 2 \cdot \frac{1}{2} + 3 \cdot 0 + 4 \cdot \frac{1}{6} = 2.$$

9 An appliance comes with a full 6-month warranty. It has been found that the time before the appliance experiences some sort of malfunction is an exponential random variable with mean 2 years.

(a) Find the exponential density function $f(x)$.

Solution: An exponential random variable has probability density function $f(x) = \lambda e^{-\lambda x}$ with $x \geq 0$. Thus we only need to find the parameter λ. Recall that an exponential random variable X has mean $E(X) = \frac{1}{\lambda}$. Since the mean in this case is 2 (years), we have $1/\lambda = 2$, or $\lambda = 1/2 = 0.5$. Thus $f(x) = 0.5e^{-x/2}$.

(b) What percentage of appliances will malfunction during the warranty period?

Solution: We are asked for $\Pr(0 \leq X \leq 0.5)$ (here 6 months is 0.5 years, and we've used years in part (a)). This is computed as follows:

$$\begin{aligned}\Pr(0 \leq X \leq 0.5) &= \int_0^{0.5} 0.5e^{-x/2}\,dx \\ &= -e^{-x/2}\Big|_0^{0.5} \\ &= -e^{-0.5/2} - \left(-e^0\right) \\ &= 1 - e^{-0.25} \\ &\approx 0.2212.\end{aligned}$$

Thus roughly 22.12% of appliances will malfunction during the first six months.

10 A farmer has observed that the time to maturation of a certain crop is approximately normally distributed with mean, μ, of 60 days and standard deviation, σ, of 2 days. Use the attached table for standard normal distribution to find the percentage of plants that will mature in less than 57 days.

Solution: We are asked to find $\Pr(X \leq 57)$, where X is a normal random variable with mean $E(X) = \mu = 60$ and standard deviation $\sigma = 2$. Using the fact that $\Pr(X \leq a) = \Pr\left(Z \leq \frac{a-\mu}{\sigma}\right)$, where Z is the standard normal distribution, we get

$$\Pr(X \leq 57) = \Pr\left(Z \leq \frac{57-60}{2}\right) = \Pr(Z \leq -1.5).$$

To compute $\Pr(Z \leq -1.5)$, we must notice two facts. First, by the symmetry of the standard normal curve, $\Pr(Z \leq -1.5) = \Pr(Z \geq +1.5)$. Next, since $\Pr(Z \geq 0) = 1/2$, we have $\Pr(Z \geq 1.5) = \frac{1}{2} - \Pr(0 \leq Z \leq 1.5)$. This we can look up in a table: $\Pr(0 \leq Z \leq 1.5)$ is the Z-value 0.4332. Thus $\Pr(Z \leq -1.5) \approx 0.5 - 0.4332 = 0.0668$. Hence only about 6.68% of plants will mature in less than 57 days.

11 A random variable X is given by the following probability table:

Outcome:	0	1	2	3
Probability:	0.5	0.1		

Unfortunately, as you can see several of the entries have been lost. There is a note, however, that $E(X) = 1$. From this information complete the probability table.

Solution: Let's fill in the blanks with variables: say $\Pr(X = 2) = x$ and $\Pr(X = 3) = y$.

Outcome:	0	1	2	3
Probability:	0.5	0.1	x	y

We know two facts: first the expected value is 1. We can compute this as

$$1 = E(X) = 0(0.5) + 1(0.1) + 2(x) + 3(y) = 0 + 0.1 + 2x + 3y \qquad \text{or} \qquad 2x + 3y = 0.9.$$

The other fact is that this is a probability distribution, so the total probability is one:

$$0.5 + 0.1 + x + y = 1 \qquad \text{or} \qquad x + y = 0.4.$$

From these two equations we find that $x = 0.3$ and $y = 0.1$.

12 Consider the random variable X with probability density function $f(x) = kx^2$, $0 \le x \le 2$.

(a) What is the value of k?

Solution: We can find the value of k by recalling that $\int_a^b f(x)\,dx$ must be 1:

$$\begin{aligned} 1 &= \int_0^2 kx^2\,dx \\ &= k\frac{1}{3}x^3\Big|_0^2 \\ &= \frac{k}{3}\left(2^3 - 0^3\right) \\ &= \frac{8k}{3}. \end{aligned}$$

Thus $k = \dfrac{3}{8}$.

(b) Find the probability $\Pr(0 \le X \le 1)$.

Solution: We can compute this directly:

$$\Pr(0 \le X \le 1) = \int_0^1 \frac{3}{8}x^2\,dx = \frac{3}{8}\cdot\frac{1}{3}x^3\Big|_0^1 = \frac{1}{8}\left(1^3 - 0^3\right) = \frac{1}{8} = 0.125.$$

(c) What is the expected value $E(X)$?

Solution: The expected value may be calculated as follows:

$$\begin{aligned} E(X) &= \int_A^B xf(x)\,dx = \int_0^2 x\cdot\frac{3}{8}x^2\,dx \\ &= \frac{3}{8}\int_0^2 x^3\,dx = \frac{3}{8}\cdot\frac{1}{4}x^4\Big|_0^2 \\ &= \frac{3}{32}\left(2^4 - 0^4\right) = \frac{3}{2}. \end{aligned}$$

(d) What is the variance $\mathrm{Var}(X)$?

Solution: We calculate the variance as follows:

$$\begin{aligned} \mathrm{Var}(X) &= \int_a^b x^2 f(x)\,dx - E(X)^2 = \int_0^2 x^2\frac{3}{8}x^2\,dx - \left(\frac{3}{2}\right)^2 \\ &= \frac{3}{8}\int_0^2 x^4\,dx - \frac{9}{4} = \frac{3}{8}\cdot\frac{1}{5}x^5\Big|_0^2 - \frac{9}{4} \\ &= \frac{3}{40}\left(2^5 - 0^5\right) - \frac{9}{4} = \frac{3}{40}\cdot 32 - \frac{9}{4} = \frac{12}{5} - \frac{9}{4} = \frac{3}{20} = 0.15. \end{aligned}$$

We could also use the other formula:

$$\begin{aligned}
\operatorname{Var}(X) &= \int_a^b (x-\mu)^2 f(x)\,dx = \int_0^2 (x-1.5)^2 \cdot \frac{3}{8}x^2\,dx \\
&= \frac{3}{8}\int_0^2 \left(x^4 - 3x^3 + 2.25x^2\right)\,dx = \frac{3}{8}\left(\frac{1}{5}x^5 - \frac{3}{4}x^4 + \frac{2.25}{3}x^3\right)\Big|_0^2 \\
&= \frac{3}{8}\left(\frac{1}{5}2^5 - \frac{3}{4}2^4 + \frac{2.25}{3}2^3\right) - 0 = \frac{3}{8}\left(\frac{32}{5} - 12 + \frac{18}{3}\right) = \frac{3}{8}\cdot\frac{2}{5} \\
&= \frac{3}{20} = 0.15,
\end{aligned}$$

as before.

13 The length of an adult of a certain species of worm is assumed to be a random variable X that is normally distributed with mean 250 mm and standard deviation 15 mm. Find the probability that an adult worm (of this species) is between 238 mm and 253 mm long. (You may use a table of Z-values.)

Solution: Here X is a normal random variable with mean $\mu = 250$ and standard deviation $\sigma = 15$. We are asked for $\Pr(238 \le X \le 253)$, which may be computed as follows:

$$\begin{aligned}
\Pr(238 \le X \le 253) &= \Pr\left(\frac{238-250}{15} \le Z \le \frac{253-250}{15}\right) \\
&= \Pr\left(-0.8 \le Z \le 0.2\right),
\end{aligned}$$

where Z is the standard normal distribution. The simplest way to compute this from the table is to write it as

$$\begin{aligned}
\Pr(238 \le X \le 253) &= \Pr\left(-0.8 \le Z \le 0.2\right) \\
&= \Pr\left(-0.8 \le Z \le 0\right) + \Pr\left(0 \le Z \le 0.2\right) \\
&= \Pr\left(0 \le Z \le 0.8\right) + \Pr\left(0 \le Z \le 0.2\right),
\end{aligned}$$

by symmetry. Looking these Z-values up on the tables, we get

$$\begin{aligned}
\Pr(238 \le X \le 253) &= \Pr\left(0 \le Z \le 0.8\right) + \Pr\left(0 \le Z \le 0.2\right) \\
&= 0.2881 + 0.0793 = 0.3674.
\end{aligned}$$

Thus the probability is roughly 36.74% that a worm is between 238 mm and 253 mm long.

14 A charge on a certain laptop battery lasts, on average, for 100 minutes. Assume the battery life is an exponential random variable. What is the probability that the battery charge will last for two hours (120 minutes)?

Solution: This asks for $\Pr(X \ge 120)$, where X is the exponential random variable that describes the life (in minutes) of a battery charge. Since X is an exponential random variable, the probability density function is $f(x) = \lambda e^{-\lambda x}$ $(0 \le x < \infty)$. This constant λ may be found by recalling that $\mu = E(X) = 1/\lambda$, so (since $E(X) = 100$), $\lambda = \frac{1}{\mu} = \frac{1}{100} = 0.01$.
Thus we may compute

$$\begin{aligned}
\Pr(X \ge 120) &= 1 - \Pr(X < 120) = 1 - \int_0^{120} 0.01e^{-0.01x}\,dx \\
&= 1 - \left[-e^{-0.01x}\right]_0^{120} = 1 - \left[\left(-e^{-0.01(120)}\right) - \left(-e^{-0.01(0)}\right)\right] \\
&= 1 - \left[-e^{-1.20} + 1\right] = e^{-1.20} \approx 0.3012.
\end{aligned}$$

Thus there is roughly a 30.1% probability that the battery will last at least two hours.

Notice that we could have calculated $\Pr(X \geq 120)$ directly. We'll do this, more for completeness than anything else:

$$\begin{aligned}\Pr(X \geq 120) = \int_{120}^{\infty} 0.01e^{-0.01x}\,dx &= \lim_{b\to\infty}\int_{120}^{b} 0.01e^{-0.01x}\,dx \\ &= \lim_{b\to\infty}\left[-e^{-0.01x}\right]_{120}^{b} = \lim_{b\to\infty}\left[-e^{-0.01(b)} - (-e^{-0.01(120)})\right] \\ &= \lim_{b\to\infty}(-e^{-0.01(b)}) + e^{-1.20} \\ &= 0 + e^{-1.20} = e^{-1.20} \approx 0.3012.\end{aligned}$$

15 Let $S = \{\ 1,\ 2,\ 3,\ 4,\ x,\ y,\ z\ \}$ be a set of 7 numbers. We are told that the mean of this set is 2, the median is 2.5, and the mode is 4. Find x, y, and z.

Solution: We have three conditions. The first, that the mean is 2, means that

$$\frac{1+2+3+4+x+y+z}{7} = 2 \qquad \text{or} \qquad x+y+z=4. \tag{1}$$

The second condition is that the median is 2.5. Since there are an odd number of terms, this means that the middle term is 2.5. Thus one of x, y, or z is 2.5. Let's say $x = 2.5$. The third condition is that the mode, or most common element, is 4. This means that 4 must appear a second time in the data, so one of y or z must be 4. Let's say $y = 4$. Now condition (1) says that

$$x+y+z=4 \qquad \text{or} \qquad 2.5+4+z=4 \qquad \text{or} \qquad z=-2.5.$$

Thus $\{\ x,\ y,\ z\ \} = \{\ 2.5,\ 4,\ -2.5\ \}$.