1 Find the solution to the differential equation $y' = y^2 x e^{x^2}$ assuming $y(0) = 1$.

Solution: This differential equation is separable. Rewrite y' as $\frac{dy}{dx}$, then multiply the equation by dx and divide by y^2 to get

$$\frac{dy}{y^2} = xe^{x^2}\ dx.$$

Now integrate:

$$\int y^{-2}\ dy = \int xe^{x^2}\ dx.$$

The left-hand integral is easier than the one on the right-hand side. For the one on the right, we make the substitution $u = x^2$, so $du = 2x\ dx$, or $x\ dx = \frac{1}{2}\ du$. Thus

$$\int xe^{x^2}\ dx = \int e^{x^2} x\ dx = \int e^u \frac{1}{2}\ du = \frac{1}{2}\int e^u\ du = \frac{1}{2}e^u + K = \frac{1}{2}e^{x^2} + K.$$

Hence, if we integrate the left-hand integral as well, we get

$$-y^{-1} = \frac{1}{2}e^{x^2} + K \qquad \text{or} \qquad y = -\frac{2}{e^{x^2} + 2K}.$$

Writing K_1 for $2K$, we find that plugging in $x = 0$, $y = 1$ yields

$$1 = -\frac{2}{e^0 + K_1} = -\frac{2}{1 + K_1}.$$

Solving, we find that $K_1 = -3$, so $y = -\dfrac{2}{e^{x^2} - 3}$.

2 Consider the differential equation $y' = 2t(y - 2)$.

(a) Find all constant solutions (that is, solutions $y = C$) of the differential equation.

Solution: The constant solutions $y = k$ are those for which the right-hand side $2t(y - 2)$ is identically zero. This occurs only when $y = 2$.

(b) For what value of k is the function $y = e^{t^2} + k$ a solution to this differential equation?

Solution: We differentiate and find that $f'(t) = 2te^{t^2}$. This means that

$$\text{right-hand side} = 2t(y - 2) = 2t\left(e^{t^2} + k - 2\right)$$

agrees with

$$\text{left-hand side} = y' = 2te^{t^2}$$

precisely when $k - 2 = 0$, or $k = 2$.

3 Let $y = Z(t)$ be the number of zebras at time t. Suppose that the rate of growth of the zebra population is proportional to the difference between 10,000 and the population at time t. When $Z(t)$ is small, the population is increasing.

(a) Write down a differential equation for $Z(t)$. State whether the proportionality constant is positive or negative.

Solution: The differential equation in question is

$$Z'(t) = k\,(10{,}000 - Z(t))$$

[Here $Z'(t)$ is "the rate of growth of the zebra population", $10,000 - Z(t)$ is "the difference between 10,000 and the population at time t", and the constant k comes from the statement "is proportional to".]

Since the population is increasing (so $Z'(t) > 0$) when $Z(t)$ is small (so $10,000 - Z(t) > 0$), the constant k must be positive. That is, $k > 0$.

(b) Suppose a new predator is introduced that kills 150 zebras per year. Modify the differential equation in (a) to account for this new information.

Solution: We assume that the new predator kills 150 zebras per year *continuously*, so that the differential equation becomes

$$Z'(t) = k\,(10{,}000 - Z(t)) - 150.$$

4 (a) Find the general solution to the following differential equation:

$$\frac{dy}{dx} = \frac{xy^2}{1+3x^2}.$$

Solution: This differential equation is separable:

$$\frac{dy}{y^2} = \frac{x}{1+3x^2}\,dx.$$

Now we integrate both sides:

$$\int y^{-2}\,dy = \int \frac{x}{1+3x^2}\,dx.$$

The integral on the right is done using the substitution $u = 1+3x^2$, $du = 6x\,dx$:

$$-y^{-1} = \frac{1}{6}\ln(1+3x^2) + K.$$

Solving for y, we get

$$y = -\frac{6}{\ln(1+3x^2) + K_1},$$

where K_1 is an unknown constant (in fact, $K_1 = 6K$, for whatever that's worth).

(b) Find the general solution to the following differential equation:

$$xy' - 3y = x^3 \qquad \text{for } x > 0.$$

Solution: This is a first-order linear differential equation, so we divide by x to put it into standard form:

$$y' - \frac{3}{x}y = x^2.$$

This has $P(x) = -3/x$, so $\int P(x)\,dx$ may be taken to be $-3\ln(x) = \ln(x^{-3})$. Thus our integrating factor is $h(x) = e^{\int P(x)\,dx} = e^{\ln(x^{-3})} = x^{-3}$, so we get

$$x^{-3}y' - 3x^{-4}y = x^{-1}$$

or

$$\left(x^{-3}y\right)' = x^{-1}.$$

Integrating both sides, we get

$$x^{-3}y = \int x^{-1}\,dx = \ln(x) + K.$$

Multiplying through by x^3 gives us the general solution:

$$y = x^3\ln(x) + Kx^3.$$

5 A boiled egg with initial temperature of 212° F is plunged into a water-filled container that is kept at a constant temperature of 40° F. The rate of change of the temperature of the egg, $f(t)$, at time t, in minutes, is proportional to the difference of the temperatures of the water and the egg.

(a) Set up an initial value problem that the is satisfied by the temperature of the cooling egg, as described above.

Solution: The initial value problem is

$$f'(t) = k\,(40 - f(t))$$
$$f(0) = 212.$$

This may be seen as follows. The key sentence is the last one in the last one in the statement of the problem:

> The rate of change of the temperature of the egg (*that is,* $f'(t)$) is proportional to (*that is, equals some constant* k *times*) the difference of the temperatures of the water and the egg (*that is,* $40 - f(t)$).

Thus $f'(t) = k\,(40 - f(t))$. The initial condition is simply that the temperature of the egg at time $t = 0$ is $212°$.

(b) Determine the sign of the constant of proportionality, with appropriate reasoning.

Solution: Since $f'(t) < 0$ (the egg is cooling, so the change in temperature is negative) and $40 - f(t) < 0$ (that is, the temperature of the egg is *greater* than $40°$), we must have $k > 0$.

Notice that an equally valid differential equation would be $f'(t) = k\,(f(t) - 40)$; in this case, k would be negative.

(c) What would be the temperature of the egg after one week?

Solution: As t grows without bound, the temperature of the egg tends to the temperature of the water surrounding it. That is, $f(t) \approx 40°$ F after 1 week. (Notice that 1 week is

$$t = (7 \text{ days})\left(\frac{24 \text{ hours}}{1 \text{ day}}\right)\left(\frac{60 \text{ minutes}}{1 \text{ hour}}\right) = 10{,}080 \text{ minutes},$$

so 1 week is really a large value for t.)

6 (a) Solve the differential equation

$$y' = t^2 e^{-y} - e^{-y}$$

Solution: This is a separable differential equation:

$$\frac{dy}{dt} = \left(t^2 - 1\right) e^{-y}.$$

We multiply the entire equation by $e^y\,dt$ to get

$$e^y\,dy = \left(t^2 - 1\right)\,dt.$$

Now we integrate both sides:

$$\int e^y\,dy = \int \left(t^2 - 1\right)\,dt$$
$$e^y = \frac{1}{3}t^3 - t + K.$$

Now we solve for y by taking the natural log of both sides:

$$y = \ln(e^y) = \ln\left(\frac{1}{3}t^3 - t + K\right).$$

Thus the general solution to this differential equation is $y = \ln\left(\frac{1}{3}t^3 - t + K\right)$.

(b) Solve the initial value problem

$$y' = 2y + e^{5t}$$
$$y(0) = 1.$$

Solution: This equation is a first-order linear differential equation. We put it in standard form:

$$y' - 2y = e^{5t}.$$

Thus $P(t) = -2$ and $G(t) = e^{5t}$. We can choose $\int P(t)\,dt = -2t$ as the "simplest" anti-derivative, so our integrating factor is $e^{\int P(t)\,dt} = e^{-2t}$. We multiply our equation by this integrating factor, and (as always) notice that the left-hand side of the equation is $(h(t)y)'$:

$$e^{-2t}y' - 2e^{-2t}y = e^{-2t} \cdot e^{5t}$$
$$\left(e^{-2t}y\right)' = e^{3t}.$$

Now we integrate both sides:

$$\int \left(e^{-2t}y\right)'\,dt = \int e^{3t}\,dt$$
$$e^{-2t}y = \frac{1}{3}e^{3t} + K.$$

Multiplying through by e^{2t} allows us to solve for y:

$$y = e^{2t}\left(\frac{1}{3}e^{3t} + K\right)$$
$$y = \frac{1}{3}e^{5t} + Ke^{2t}.$$

Now we solve for K by plugging in the initial condition $y(0) = 1$ (or $y = 1$ when $t = 0$):

$$1 = \frac{1}{3}e^{5\cdot 0} + Ke^{2\cdot 0} = \frac{1}{3} + K.$$

Thus $K = \frac{2}{3}$, so our final answer is $y = \dfrac{1}{3}e^{5t} + \dfrac{2}{3}e^{2t}$.

7 A cup of coffee cools on a table. The coffee, when it is first placed on the table, is 170° F. After 5 minutes, the coffee has cooled to 160° F. The room temperature is a constant 70° F. Assume that the rate of change of the temperature of the coffee is proportional to the difference between the temperature of the room and the temperature of the coffee.

(a) Set up an initial value problem that is satisfied by the temperature of the cooling cup of coffee. **You do not need to solve this equation!**

Solution:

Let $T(t)$ be the temperature (in degrees Fahrenheit) of the coffee t minutes after it is placed on the table. Then the sentence

Assume that the rate of change of the temperature of the coffee is proportional to the difference between the temperature of the room and the temperature of the coffee.

means that

$$T'(t) \propto (70 - T(t)).$$

This means that for some constant k (the constant of proportionality),

$$T'(t) = k\,(70 - T(t)).$$

The initial value problem that is asked for is this differential equation together with the initial condition $T(0) = 170$ (the temperature at time $t = 0$ is 170° F). Thus the final answer is

$$T'(t) = k\,(70 - T(t))$$
$$T(0) = 170.$$

The other condition, that $T(5) = 160$, is not part of the initial value problem, but it would allow one to find the value of k.

(b) Determine the sign of the constant of proportionality in your equation from part (a). Explain your reasoning!

Solution: One can actually solve for k, but that requires a fair amount of work. It's simpler to notice two facts:

- $T'(t) < 0$, since the temperature of the coffee is decreasing, and
- $70 - T(t) < 0$, since the temperature of the coffee is above room temperature.

Thus both $T'(t)$ and $70 - T(t)$ are negative, so the constant k must be positive.

Remarks: Notice that, in part (a), one *could* have used the equation

$$T'(t) = k\,(T(t) - 70)\,.$$

Now the left-hand side is still negative, but the difference term on the right is positive, so k must be negative. That is, the answer to part (b) depends on the form of the answer to part (a).

8 Suppose on this campus of 9952 students, one student returns from break with the flu virus and proceeds to share that virus with the student population. Assume that the virus spreads at a rate proportional to the product of the number of students presently infected and the number of students not yet infected. Determine the appropriate differential equation that models this situation. *Please do not try to solve this differential equation.*

Solution: Let $f(t)$ be the number of students infected with the flu virus at time t. The problem says that the rate of change of $f(t)$ is proportional to the product of the number of infected and the number of uninfected students at time t. Since there are 9952 students on campus, at time t there are $9952 - f(t)$ uninfected students. So we reformulate this problem to say

$$f'(t) \propto f(t)\,(9952 - f(t))$$

or

$$f'(t) = kf(t)\,(9952 - f(t))$$

for some constant of proportionality k.

This equation is tricky to solve, which is why we don't ask you to solve it. It is separable, and the general solution is

$$y = \frac{NCe^{Nkt}}{1 + Ce^{Nkt}}$$

where C is an arbitrary constant, $N = 9952$, and k is the constant of proportionality.

9 (a) Write the system

$$\begin{aligned} x + y + z &= 3 \\ x - y + z &= 7 \\ x - y - z &= 1 \end{aligned}$$

in $AX = B$ format.

Solution: This is the system

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & -1 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 \\ 7 \\ 1 \end{bmatrix}.$$

That is,

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & -1 & -1 \end{bmatrix}, \qquad X = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \qquad \text{and} \qquad B = \begin{bmatrix} 3 \\ 7 \\ 1 \end{bmatrix}.$$

(b) Rewrite the system in the augmented matrix format $[\ A \mid B\ ]$.

Solution: The augmented matrix is

$$[A \mid B] = \left[\begin{array}{ccc|c} 1 & 1 & 1 & 3 \\ 1 & -1 & 1 & 7 \\ 1 & -1 & -1 & 1 \end{array}\right].$$

(c) Use the method of Gaussian elimination to solve for x, y, and z, if they exist.

Solution: We perform the Gaussian elimination below. We will use the notation $R_2 = r_2 - r_1$ to mean "Row 2 minus Row 1 is the new Row 2" in our notes to the right:

$$\begin{aligned}
[A \mid B] &= \left[\begin{array}{ccc|c} 1 & 1 & 1 & 3 \\ 1 & -1 & 1 & 7 \\ 1 & -1 & -1 & 1 \end{array}\right] \\
&\to \left[\begin{array}{ccc|c} 1 & 1 & 1 & 3 \\ 0 & -2 & 0 & 4 \\ 0 & -2 & -2 & -2 \end{array}\right] && R_2 = r_2 - r_1,\ R_3 = r_3 - r_1 \\
&\to \left[\begin{array}{ccc|c} 1 & 1 & 1 & 3 \\ 0 & 1 & 0 & -2 \\ 0 & -2 & -2 & -2 \end{array}\right] && R_2 = \frac{1}{-2} r_2 \\
&\to \left[\begin{array}{ccc|c} 1 & 1 & 1 & 3 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & -2 & -6 \end{array}\right] && R_3 = r_3 + 2r_2 \\
&\to \left[\begin{array}{ccc|c} 1 & 1 & 1 & 3 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & 3 \end{array}\right] && R_3 = \frac{1}{-2} r_3.
\end{aligned}$$

Now we use back substitution. That is, the above augmented matrix represents the upper triangular system

$$\begin{aligned} x + y + z &= 3 \\ y \quad\ &= -2 \\ z &= 3. \end{aligned}$$

Working backwards, this gives us $z = 3$, $y = -2$, and $x + (-2) + (3) = 3$, or $x = 2$. That is, $(x, y, z) = (2, -2, 3)$.

10 Suppose the graph of a quadratic equation $y = ax^2 + bx + c$ passes through the points $(1, 5)$, $(2, 13)$, and $(3, 25)$. Find a, b, c.

Solution: To find a, b, and c, we need some equations that these constants satisfy. These equations are obtained by plugging in the (x, y) points on the quadratic. For example, if we plug in $x = 1$ and $y = 5$, we get

$$5 = a(1)^2 + b(1) + c.$$

Doing this three times gives us the three equations (written "backwards" for reasons that I hope will become clear):

$$\begin{aligned} c + b + a &= 5 \\ c + 2b + 4a &= 13 \\ c + 3b + 9a &= 25. \end{aligned}$$

To solve this, we'll use elimination (row reduction) on the augmented matrix

$$[A\,|\,B] = \left[\begin{array}{ccc|c} 1 & 1 & 1 & 5 \\ 1 & 2 & 4 & 13 \\ 1 & 3 & 9 & 25 \end{array}\right] \longrightarrow \left[\begin{array}{ccc|c} 1 & 1 & 1 & 5 \\ 0 & 1 & 3 & 8 \\ 0 & 2 & 8 & 20 \end{array}\right] \qquad R_2 = r_2 - r_1, R_3 = r_3 - r_1$$

$$\longrightarrow \left[\begin{array}{ccc|c} 1 & 1 & 1 & 5 \\ 0 & 1 & 3 & 8 \\ 0 & 0 & 2 & 4 \end{array}\right] \qquad R_3 = r_3 - 2r_2$$

$$\longrightarrow \left[\begin{array}{ccc|c} 1 & 1 & 1 & 5 \\ 0 & 1 & 3 & 8 \\ 0 & 0 & 1 & 2 \end{array}\right] \qquad R_3 = \frac{1}{2}r_3.$$

(Now I hope it's clear why we looked at the coefficients in reverse order – it made it simpler to row reduce!) Thus we have the system

$$\begin{aligned} c + b + a &= 5 \\ b + 3a &= 8 \\ a &= 2 \end{aligned}$$

from which we get $a = 2$, $b = 2$ (since $b + 3(2) = 8$), and $c = 1$ (since $c + 2 + 2 = 5$). Thus the quadratic equation we're looking for is $y = 2x^2 + 2x + 1$.

11 Find all the values of α and β so that the system

$$\begin{aligned} x + 3y - 2z &= 5 \\ 2x + 4y + 3z &= -3 \\ -x + y + \alpha z &= \beta \end{aligned}$$

(a) ... has a unique solution.
(b) ... has no solution.
(c) ... has infinitely many solutions.

Solution: Let's perform elimination (row reduction) on the augmented matrix

$$[A\,|\,B] = \left[\begin{array}{ccc|c} 1 & 3 & -2 & 5 \\ 2 & 4 & 3 & -3 \\ -1 & 1 & \alpha & \beta \end{array}\right] \longrightarrow \left[\begin{array}{ccc|c} 1 & 3 & -2 & 5 \\ 0 & -2 & 7 & -13 \\ 0 & 4 & \alpha-2 & \beta+5 \end{array}\right] \qquad R_2 = r_2 - 2r_1, R_3 = r_3 + r_1$$

$$\longrightarrow \left[\begin{array}{ccc|c} 1 & 3 & -2 & 5 \\ 0 & 1 & -3.5 & 6.5 \\ 0 & 4 & \alpha-2 & \beta+5 \end{array}\right] \qquad R_2 = \frac{1}{2}r_2$$

$$\longrightarrow \left[\begin{array}{ccc|c} 1 & 3 & -2 & 5 \\ 0 & 1 & -3.5 & 6.5 \\ 0 & 0 & \alpha+12 & \beta-21 \end{array}\right] \qquad R_3 = r_3 - 4r_2.$$

This will have a unique solution when $\alpha + 12 \neq 0$, or $\alpha \neq -12$. If $\alpha = -12$, then it has no solution if $\beta - 21 \neq 0$, or $\beta \neq 21$. In the final case (when $\alpha = -12$ and $\beta = 21$), there are infinitely many solutions.

12 Find the inverses of the following matrices, if these inverses exist:

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 1 \\ 2 & 4 & 9 \end{bmatrix} \qquad B = \begin{bmatrix} 1 & 2 & 3 \\ 6 & 7 & 8 \\ 9 & 17 & 0 \end{bmatrix} \qquad C = \begin{bmatrix} 2 & 4 & 0 & -2 & 0 \\ 3 & 6 & 0 & -4 & 1 \\ 5 & 10 & 1 & -4 & 0 \\ 7 & 14 & 2 & -5 & 0 \\ 4 & 8 & 3 & -2 & 5 \end{bmatrix}$$

Solution:

(a) $A^{-1} = \begin{bmatrix} 5/3 & -2 & -1/3 \\ 2/3 & 1 & -1/3 \\ -2/3 & 0 & 1/3 \end{bmatrix} = \dfrac{1}{3}\begin{bmatrix} 5 & -6 & -1 \\ 2 & 3 & -1 \\ -2 & 0 & 1 \end{bmatrix}.$

We find this inverse by elimination (row reduction). We start with $[\,A\,|\,I\,]$ and reduce to $[\,I\,|\,A^{-1}\,]$:

$$[\,A\,|\,I\,] = \left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 2 & 4 & 9 & 0 & 0 & 1 \end{array}\right] \longrightarrow \left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 3 & -2 & 0 & 1 \end{array}\right] \qquad R_3 = r_3 - 2r_1$$

$$\longrightarrow \left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & -2/3 & 0 & 1/3 \end{array}\right] \qquad R_3 = \frac{1}{3}r_3$$

$$\longrightarrow \left[\begin{array}{ccc|ccc} 1 & 2 & 0 & 3 & 0 & -1 \\ 0 & 1 & 0 & 2/3 & 1 & -1/3 \\ 0 & 0 & 1 & -2/3 & 0 & 1/3 \end{array}\right] \qquad R_1 = r_1 - 3r_3, R_2 = r_2 - r_1$$

$$\longrightarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 5/3 & -2 & -1/3 \\ 0 & 1 & 0 & 2/3 & 1 & -1/3 \\ 0 & 0 & 1 & -2/3 & 0 & 1/3 \end{array}\right] \qquad R_1 = r_1 - 2r_2.$$

This is now $[\,I\,|\,A^{-1}\,]$, so we can read the answer from the right-hand side.

(b) $B^{-1} = \begin{bmatrix} -1.088 & 0.408 & -0.04 \\ 0.576 & -0.216 & 0.08 \\ 0.312 & 0.008 & -0.04 \end{bmatrix} = \dfrac{1}{125}\begin{bmatrix} -136 & 51 & -5 \\ 72 & -27 & 10 \\ 39 & 1 & -5 \end{bmatrix}.$

We find this inverse by elimination (row reduction). We start with $[\,B\,|\,I\,]$ and reduce to $[\,I\,|\,B^{-1}\,]$, as in part (a):

$$[\,B\,|\,I\,] = \left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 6 & 7 & 8 & 0 & 1 & 0 \\ 9 & 17 & 0 & 0 & 0 & 1 \end{array}\right] \longrightarrow \left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & -5 & -10 & -6 & 1 & 0 \\ 0 & -1 & -27 & -9 & 0 & 1 \end{array}\right] \qquad R_2 = r_2 - 6r_1, R_3 = r_3 - 9r_1$$

$$\longrightarrow \left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 1 & 2 & 1.2 & -0.2 & 0 \\ 0 & -1 & -27 & -9 & 0 & 1 \end{array}\right] \qquad R_2 = \frac{1}{-5}r_2$$

$$\longrightarrow \left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 1 & 2 & 1.2 & -0.2 & 0 \\ 0 & 0 & -25 & -7.8 & -0.2 & 1 \end{array}\right] \qquad R_3 = r_3 + r_2$$

$$\longrightarrow \left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 1 & 2 & 1.2 & -0.2 & 0 \\ 0 & 0 & 1 & 0.312 & 0.008 & -0.04 \end{array}\right] \qquad R_3 = -\frac{1}{25}r_3$$

$$\longrightarrow \left[\begin{array}{ccc|ccc} 1 & 2 & 0 & 0.064 & -0.024 & 0.12 \\ 0 & 1 & 0 & 0.576 & -0.216 & 0.08 \\ 0 & 0 & 1 & 0.312 & 0.008 & -0.04 \end{array}\right] \qquad R_1 = r_1 - 3r_3, R_2 = r_2 - 2r_3$$

$$\longrightarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -1.088 & 0.408 & -0.04 \\ 0 & 1 & 0 & 0.576 & -0.216 & 0.08 \\ 0 & 0 & 1 & 0.312 & 0.008 & -0.04 \end{array}\right] \qquad R_1 = r_1 - 2r_2.$$

This is $[\,I\,|\,B^{-1}\,]$. (And ick! Whose idea was this?)

(c) C is not invertible.

When we try to row reduce from $[\,C\,|\,I\,]$ to $[\,I\,|\,C^{-1}\,]$, we very quickly get to a position where it is clear we will be unable to succeed. Let's see:

$$[\,C\,|\,I\,] = \left[\begin{array}{rrrrr|rrrrr} 2 & 4 & 0 & -2 & 0 & 1 & 0 & 0 & 0 & 0 \\ 3 & 6 & 0 & -4 & 1 & 0 & 1 & 0 & 0 & 0 \\ 5 & 10 & 1 & -4 & 0 & 0 & 0 & 1 & 0 & 0 \\ 7 & 14 & 2 & -5 & 0 & 0 & 0 & 0 & 1 & 0 \\ 4 & 8 & 3 & -2 & 5 & 0 & 0 & 0 & 0 & 1 \end{array}\right]$$

$$\longrightarrow \left[\begin{array}{rrrrr|rrrrr} 1 & 2 & 0 & -1 & 0 & 0.5 & 0 & 0 & 0 & 0 \\ 3 & 6 & 0 & -4 & 1 & 0 & 1 & 0 & 0 & 0 \\ 5 & 10 & 1 & -4 & 0 & 0 & 0 & 1 & 0 & 0 \\ 7 & 14 & 2 & -5 & 0 & 0 & 0 & 0 & 1 & 0 \\ 4 & 8 & 3 & -2 & 5 & 0 & 0 & 0 & 0 & 1 \end{array}\right] \quad R_1 = r_1/2$$

$$\longrightarrow \left[\begin{array}{rrrrr|rrrrr} 1 & 2 & 0 & -1 & 0 & 0.5 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -4 & 1 & -1.5 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & -4 & 0 & -2.5 & 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & -5 & 0 & -3.5 & 0 & 0 & 1 & 0 \\ 0 & 0 & 3 & -2 & 5 & -2 & 0 & 0 & 0 & 1 \end{array}\right] \quad \begin{array}{l} R_2 = r_2 - 3r_1 \\ R_3 = r_3 - 5r_1 \\ R_4 = r_4 - 7r_1 \\ R_5 = r_5 - 4r_1 \end{array}$$

We should already be able to see that we're doomed to fail. We'll never be able to turn the left-hand side of the matrix into the identity matrix – there is no way we'll get a 1 in the second column without a $1/2$ in the first column, for example. Thus C is not invertible.

13 Which of the following 3×3 matrices are invertible? For those that are invertible, find their inverse and use it to solve the linear system $AX = B$, where

$$X = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 3 \\ -1 \\ 2 \end{bmatrix}.$$

(a) $A = \begin{bmatrix} 1 & -4 & -6 \\ 0 & 1 & 4 \\ 0 & 0 & 2 \end{bmatrix}$

Solution: $A^{-1} = \begin{bmatrix} 1 & 4 & -5 \\ 0 & 1 & -2 \\ 0 & 0 & 1/2 \end{bmatrix}$ and so $X = A^{-1}B = \begin{bmatrix} 1 & 4 & -5 \\ 0 & 1 & -2 \\ 0 & 0 & 1/2 \end{bmatrix} \begin{bmatrix} 3 \\ -1 \\ 2 \end{bmatrix} = \begin{bmatrix} -11 \\ -5 \\ 1 \end{bmatrix}$.

We'll show that the inverse of A is what we say it is by row reducing $[\,A\,|\,I\,]$ into $[\,I\,|\,A^{-1}\,]$:

$$[\,A\,|\,I\,] = \left[\begin{array}{rrr|rrr} 1 & -4 & -6 & 1 & 0 & 0 \\ 0 & 1 & 4 & 0 & 1 & 0 \\ 0 & 0 & 2 & 0 & 0 & 1 \end{array}\right] \longrightarrow \left[\begin{array}{rrr|rrr} 1 & -4 & -6 & 1 & 0 & 0 \\ 0 & 1 & 4 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1/2 \end{array}\right] \quad R_3 = \frac{1}{2} r_3$$

$$\longrightarrow \left[\begin{array}{rrr|rrr} 1 & -4 & 0 & 1 & 0 & 3 \\ 0 & 1 & 0 & 0 & 1 & -2 \\ 0 & 0 & 1 & 0 & 0 & 1/2 \end{array}\right] \quad R_1 = r_1 + 6r_3, R_2 = r_2 - 4r_3$$

$$\longrightarrow \left[\begin{array}{rrr|rrr} 1 & 0 & 0 & 1 & 4 & -5 \\ 0 & 1 & 0 & 0 & 1 & -2 \\ 0 & 0 & 1 & 0 & 0 & 1/2 \end{array}\right] \quad R_1 = r_1 + 4r_2.$$

This is $[\,I\,|\,A^{-1}\,]$, so we can read A^{-1} off the right-hand side. It is what we claimed.

(b) $A = \begin{bmatrix} 1 & 5 & 6 \\ 0 & 1 & 2 \\ -1 & -3 & -2 \end{bmatrix}$

Solution: This A is not invertible. When we try to do row reduce $[\,A\,|\,I\,]$ into $[\,I\,|\,A^{-1}\,]$ (as we did in part (a)), we fail:

$$[\,A\,|\,I\,] = \left[\begin{array}{ccc|ccc} 1 & 5 & 6 & 1 & 0 & 0 \\ 0 & 1 & 2 & 0 & 1 & 0 \\ -1 & -3 & -2 & 0 & 0 & 1 \end{array}\right] \longrightarrow \left[\begin{array}{ccc|ccc} 1 & 5 & 6 & 1 & 0 & 0 \\ 0 & 1 & 2 & 0 & 1 & 0 \\ 0 & 2 & 4 & 1 & 0 & 1 \end{array}\right] \qquad R_3 = r_3 + r_1$$

$$\longrightarrow \left[\begin{array}{ccc|ccc} 1 & 5 & 6 & 1 & 0 & 0 \\ 0 & 1 & 2 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & -2 & 1 \end{array}\right] \qquad R_3 = r_3 - 2r_2.$$

We can see now that we won't be able to reduce this to $[\,I\,|\,A^{-1}\,]$ as there's no way to get the identity matrix on the left-hand side. Thus A is not invertible.

(c) $A = \begin{bmatrix} 0 & 3 & 8 \\ -1 & 0 & 2 \\ 0 & 1 & 3 \end{bmatrix}$

Solution: $A^{-1} = \begin{bmatrix} -2 & -1 & 6 \\ 3 & 0 & -8 \\ -1 & 0 & 3 \end{bmatrix}$ and so $X = A^{-1}B = \begin{bmatrix} -2 & -1 & 6 \\ 3 & 0 & -8 \\ -1 & 0 & 3 \end{bmatrix} \begin{bmatrix} 3 \\ -1 \\ 2 \end{bmatrix} = \begin{bmatrix} 7 \\ -7 \\ 3 \end{bmatrix}$.

We'll show that the inverse of A is what we say it is by row reducing $[\,A\,|\,I\,]$ into $[\,I\,|\,A^{-1}\,]$:

$$[\,A\,|\,I\,] = \left[\begin{array}{ccc|ccc} 0 & 3 & 8 & 1 & 0 & 0 \\ -1 & 0 & 2 & 0 & 1 & 0 \\ 0 & 1 & 3 & 0 & 0 & 1 \end{array}\right] \longrightarrow \left[\begin{array}{ccc|ccc} -1 & 0 & 2 & 0 & 1 & 0 \\ 0 & 3 & 8 & 1 & 0 & 0 \\ 0 & 1 & 3 & 0 & 0 & 1 \end{array}\right] \qquad R_1 \leftrightarrow R_2$$

$$\longrightarrow \left[\begin{array}{ccc|ccc} 1 & 0 & -2 & 0 & -1 & 0 \\ 0 & 1 & 3 & 0 & 0 & 1 \\ 0 & 3 & 8 & 1 & 0 & 0 \end{array}\right] \qquad R_1 = -r_1, R_2 \leftrightarrow R_3$$

$$\longrightarrow \left[\begin{array}{ccc|ccc} 1 & 0 & -2 & 0 & -1 & 0 \\ 0 & 1 & 3 & 0 & 0 & 1 \\ 0 & 0 & -1 & 1 & 0 & -3 \end{array}\right] \qquad R_3 = r_3 - 3r_2$$

$$\longrightarrow \left[\begin{array}{ccc|ccc} 1 & 0 & -2 & 0 & -1 & 0 \\ 0 & 1 & 3 & 0 & 0 & 1 \\ 0 & 0 & 1 & -1 & 0 & 3 \end{array}\right] \qquad R_3 = -r_3$$

$$\longrightarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -2 & -1 & 6 \\ 0 & 1 & 0 & 3 & 0 & -8 \\ 0 & 0 & 1 & -1 & 0 & 3 \end{array}\right] \qquad R_1 = r_1 + 2r_3, R_2 = r_2 - 3r_3.$$

This is $[\,I\,|\,A^{-1}\,]$, so A^{-1} is as we claimed.

14 Solve the system

$$\begin{array}{rcrcrcrcr} 2x_1 & - & x_2 & + & 5x_3 & - & 3x_4 & = & 1 \\ 2x_1 & - & x_2 & + & x_3 & - & x_4 & = & -1 \\ -3x_1 & + & 4x_2 & & & - & x_4 & = & 0 \end{array}$$

by using row operations to put the augmented matrix $[\ A\ |\ B\ ]$ in reduced row echelon form.

Solution: The augmented matrix for this system is

$$[\,A\,|\,B\,] = \left[\begin{array}{rrrr|r} 2 & -1 & 5 & -3 & 1 \\ 2 & -1 & 1 & -1 & -1 \\ -3 & 4 & 0 & -1 & 0 \end{array}\right].$$

We'll solve the system by putting this augmented matrix in reduced row echelon form:

$$[\,A\,|\,B\,] = \left[\begin{array}{rrrr|r} 2 & -1 & 5 & -3 & 1 \\ 2 & -1 & 1 & -1 & -1 \\ -3 & 4 & 0 & -1 & 0 \end{array}\right] \longrightarrow \left[\begin{array}{rrrr|r} 1 & -0.5 & 2.5 & -1.5 & 0.5 \\ 2 & -1 & 1 & -1 & -1 \\ -3 & 4 & 0 & -1 & 0 \end{array}\right] \qquad R_1 = \frac{1}{2}r_1$$

$$\longrightarrow \left[\begin{array}{rrrr|r} 1 & -0.5 & 2.5 & -1.5 & 0.5 \\ 0 & 0 & -4 & 2 & -2 \\ 0 & 2.5 & 7.5 & -5.5 & 1.5 \end{array}\right] \qquad R_2 = r_2 - 2r_1, R_3 = r_3 + 3r_1$$

$$\longrightarrow \left[\begin{array}{rrrr|r} 1 & -0.5 & 2.5 & -1.5 & 0.5 \\ 0 & 2.5 & 7.5 & -5.5 & 1.5 \\ 0 & 0 & -4 & 2 & -2 \end{array}\right] \qquad R_2 \leftrightarrow R_3$$

$$\longrightarrow \left[\begin{array}{rrrr|r} 1 & -0.5 & 2.5 & -1.5 & 0.5 \\ 0 & 1 & 3 & -2.2 & 0.6 \\ 0 & 0 & 1 & -0.5 & 0.5 \end{array}\right] \qquad R_2 = \frac{1}{2.5}r_2, R_3 = -\frac{1}{4}r_3$$

$$\longrightarrow \left[\begin{array}{rrrr|r} 1 & -0.5 & 0 & -0.25 & -0.75 \\ 0 & 1 & 0 & -0.7 & -0.9 \\ 0 & 0 & 1 & -0.5 & 0.5 \end{array}\right] \qquad R_1 = r_1 - 2.5r_3, R_2 = r_2 - 3r_3$$

$$\longrightarrow \left[\begin{array}{rrrr|r} 1 & 0 & 0 & -0.6 & -1.2 \\ 0 & 1 & 0 & -0.7 & -0.9 \\ 0 & 0 & 1 & -0.5 & 0.5 \end{array}\right] \qquad R_1 = r_1 + 0.5r_2.$$

This corresponds to the system

$$\begin{aligned} x_1 \qquad\qquad - 0.6x_4 &= -1.2 \\ x_2 \qquad - 0.7x_4 &= -0.9 \\ x_3 - 0.5x_4 &= 0.5 \end{aligned}$$

which has solution

$$\begin{aligned} x_1 &= -1.2 + 0.6x_4 \\ x_2 &= -0.9 + 0.7x_4 \\ x_3 &= 0.5 + 0.5x_4, \end{aligned}$$

where x_4 is arbitrary.