

Definition

A random variable X is a rule (or function) that assigns a number to each possible outcome of an experiment.

Notation:

$P(X = k)$ = the probability that the outcome is equal to k

Binomial Probabilities:

$$P(X = k) = b(n, k; p) = \binom{n}{k} p^k q^{n-k}$$

- 1 Flip a fair coin three times. Suppose we're interested in the total numbers of heads that appear. A random variable X is a way to transform the sample space

$$S = \{HHH, HHT, HTH, HTT, THH, THT, TTH, TTT\}$$

into a set of numbers $\{0, 1, 2, 3\}$.

- (a) Write down an assignment from S to $\{0, 1, 2, 3\}$. Notation: $X(e_1) = X(HHH) = 3$.
 (b) Probability assignments: $P(X = 0) = P(X = 3) = \frac{1}{8}$ and $P(X = 1) = P(X = 2) = \frac{3}{8}$.

- 2 An urn contains 3 red balls and 7 green balls.

- (a) Three balls are drawn (with replacement). Let X be the random variable of the number of green balls selected. Write down a probability table and find $E(X)$.
 (b) Three balls are drawn (now without replacement). Let Y be the random variable of the number of green balls selected. Write down a probability table and find $E(Y)$.

- 3 You take a True-False test and guess on each of the three questions. Let X be the number of right answers minus the number of wrong answers. Find X and write down a probability table. What is the expected value $E(X)$?

- 4 You want to give a Multiple Choice test (each with 5 possible answers) with three questions. You'll give one point for each correct answer, but (so that guessing has no value) you'll also deduct some number of points for wrong (or missing) answers. Let k be the number of points lost for a wrong answer, and let X be the number of right answers minus k times the number of wrong answers. Find X and write down a probability table. What is k so that the expected value $E(X)$ is zero?

- 2 (a) If we think of this as a Bernoulli Trial (with a green ball being a “success”) then it is easy to see that the probability table should be as follows:

X	$P(X = k)$
0	$b(3, 0; 0.7) = (0.3)^3 = 0.027$
1	$b(3, 1; 0.7) = \binom{3}{1}(0.7)(0.3)^2 = 0.189$
2	$b(3, 2; 0.7) = \binom{3}{2}(0.7)^2(0.3) = 0.441$
3	$b(3, 3; 0.7) = (0.7)^3 = 0.343$

Thus $E(X)$, the expected number of green balls drawn, should be

$$\begin{aligned} E(X) &= 0(0.027) + 1(0.189) + 2(0.441) + 3(0.343) \\ &= 2.100. \end{aligned}$$

This is what we should expect: this is a Bernoulli trial, so the expected value should be $E = n \cdot p = 3 \cdot 0.7 = 2.1$.

- (b) Here the computation of the probabilities is more complicated. Probabably the simplest approach is to draw a tree. In any case, the probabilities work out to be:

X	$P(X = k)$
0	$\frac{3}{10} \cdot \frac{2}{9} \cdot \frac{1}{8} = \frac{1}{120}$
1	$\frac{7}{10} \cdot \frac{3}{9} \cdot \frac{2}{8} + \frac{3}{10} \cdot \frac{7}{9} \cdot \frac{2}{8} + \frac{3}{10} \cdot \frac{2}{9} \cdot \frac{7}{8} = \frac{7}{40}$
2	$\frac{7}{10} \cdot \frac{6}{9} \cdot \frac{3}{8} + \frac{7}{10} \cdot \frac{3}{9} \cdot \frac{6}{8} + \frac{3}{10} \cdot \frac{7}{9} \cdot \frac{6}{8} = 3 \cdot \frac{7 \cdot 6 \cdot 3}{10 \cdot 9 \cdot 8} = \frac{21}{40}$
3	$\frac{7}{10} \cdot \frac{6}{9} \cdot \frac{5}{8} = \frac{7}{24}$

Thus the expected value $E(X)$ is

$$\begin{aligned} E(X) &= 0 \cdot \frac{1}{120} + 1 \cdot \frac{7}{40} + 2 \cdot \frac{21}{40} + 3 \cdot \frac{7}{24} \\ &= \frac{84}{40} = 2.1. \end{aligned}$$

- 3 Here it is easiest to think of X as a “score” – the number right minus the number wrong. The number correct follows a binomial distribution (it’s a Bernoulli Trial since we’re just guessing). Thus we have the following probability table:

Number correct	X	$P(X = k)$
3	3	$b(3, 3; 0.5) = \frac{1}{8}$
2	1	$b(3, 2; 0.5) = \frac{3}{8}$
1	-1	$b(3, 1; 0.5) = \frac{3}{8}$
0	-3	$b(3, 0; 0.5) = \frac{1}{8}$

Thus the expected value $E(X)$ is

$$E(X) = 3 \cdot \frac{1}{8} + 1 \cdot \frac{3}{8} + (-1) \cdot \frac{3}{8} + (-3) \cdot \frac{1}{8} = 0.$$

- 4 We do this much as problem 3, above:

Number correct	X	$P(X = k)$
3	3	$b(3, 3; \frac{1}{5}) = \left(\frac{1}{5}\right)^3 = \frac{1}{125}$
2	$2 - k$	$b(3, 2; \frac{1}{5}) = \binom{3}{2} \left(\frac{1}{5}\right)^2 \left(\frac{4}{5}\right)^1 = \frac{12}{125}$
1	$1 - 2k$	$b(3, 1; \frac{1}{5}) = \binom{3}{1} \left(\frac{1}{5}\right)^1 \left(\frac{4}{5}\right)^2 = \frac{48}{125}$
0	$0 - 3k$	$b(3, 0; \frac{1}{5}) = \left(\frac{4}{5}\right)^3 = \frac{64}{125}$

Thus the expected value $E(X)$ is

$$E(X) = 3 \cdot \frac{1}{125} + (2 - k) \cdot \frac{12}{125} + (1 - 2k) \cdot \frac{48}{125} + (-3k) \cdot \frac{64}{125} = 0.$$

Multiplying through by 125, we get

$$3 + 12(2 - k) + 48(1 - 2k) + 64(-3k) = 0,$$

or

$$3 + 24 - 12k + 48 - 96k - 192k = 0,$$

or $75 = 300k$, so $k = \frac{1}{4}$.

This is what we expect. The expected number right is $n \cdot p = 3 \cdot \frac{1}{5} = \frac{3}{5}$ and the expected number wrong is $n \cdot (1 - p) = 3 \cdot \frac{4}{5} = \frac{12}{5}$. Thus $E(X) = \frac{3}{5} - k \cdot \frac{12}{5} = 0$, so $k = \frac{1}{4}$.