

1 Suppose $f(x, y) = x^3 + 3y^2 - 12xy + 13$.

(a) Find all the critical points of the function $f(x, y)$.

Solution: The critical points are the points (x, y) where both $f_x = 0$ and $f_y = 0$. We calculate and find that

$$f_x = 3x^2 - 12y \quad \text{and} \quad f_y = 6y - 12x.$$

From $f_y = 0$ we see that $6y - 12x = 0$ or $y = 2x$. Plugging this into $f_x = 0$, we get

$$0 = 3x^2 - 12y = 3x^2 - 12 \cdot 2x = 3x^2 - 24x = 3x(x - 8).$$

Thus $x = 0$ or $x = 8$. We can find the y -coordinates of our critical points by simply plugging in these numbers into $y = 2x$. Thus our critical points are $(x, y) = (0, 0)$ and $(x, y) = (8, 16)$.

(b) Classify each of the critical points you found in part (a) as a local maximum, a local minimum, or a saddle point. Recall that the tests are: at the critical point,

- if $D > 0$ and $f_{xx} < 0$, then the critical point is a local maximum;
- if $D > 0$ and $f_{xx} > 0$, then the critical point is a local minimum;
- if $D < 0$, then the critical point is a saddle point.

Solution: The second derivatives for our function $f(x, y)$ are

$$f_{xx} = 6x, \quad f_{xy} = f_{yx} = -12, \quad \text{and} \quad f_{yy} = 6.$$

Thus $D = f_{xx}f_{yy} - f_{xy}^2$ is $D = (6x)(6) - (-12)^2 = 36x - 144$. At the point $(x, y) = (0, 0)$, we therefore have $D = 36(0) - 144 = -144 < 0$, so $(0, 0)$ is a saddle point (by the test given above). At the other critical point $(x, y) = (8, 16)$, the value of D is $D = 36(8) - 144 = 144 > 0$, so this critical point is either a maximum or a minimum, depending on the sign of f_{xx} at the critical point. Since $f_{xx} = 6x = 6 \cdot 8 > 0$, this critical point is a local minimum.

2 A company is making two products. The demand functions are given by

$$p_1 = 20 - x \quad \text{and} \quad p_2 = 24 - 2y$$

where x and y are given in thousands of units and the prices p_1 and p_2 are in thousands of dollars. The joint cost function is given by $C(x, y) = x^2 + 4xy + 2y^2$. Find the maximum possible profit.

Hint: Recall that profit is revenue minus cost: $P = R - C$.

Solution: Since we're given the cost function, we need only find the revenue function on our way to finding the profit. The revenue from selling the first product is the x , number of units sold, times p_1 , the price of each unit. Similarly, the revenue from the second product is y times p_2 , so the revenue function is

$$R(x, y) = xp_1 + yp_2 = x(20 - x) + y(24 - 2y) = 20x - x^2 + 24y - 2y^2.$$

Thus the profit function that we wish to maximize is

$$\begin{aligned} P(x, y) &= R(x, y) - C(x, y) \\ &= (20x - x^2 + 24y - 2y^2) - (x^2 + 4xy + 2y^2) \\ &= 20x - 2x^2 - 4xy + 24y - 4y^2. \end{aligned}$$

We maximize this function in the usual way, by setting the first partial derivatives equal to zero:

$$P_x = 20 - 4x - 4y = 0 \quad \text{and} \quad P_y = -4x + 24 - 8y = 0.$$

This linear system simplifies to

$$\begin{aligned} 4x + 4y &= 20 \\ 4x + 8y &= 24 \end{aligned}$$

which has solution (critical point) $(x, y) = (4, 1)$. At this point the profit is

$$P(4, 1) = 20(4) - 2(4)^2 - 4(4)(1) + 24(1) - 4(1)^2 = 52,$$

or \$52,000.

We can check that this is a maximum (so that we haven't just minimized our profits!) by examining the second partial derivatives:

$$P_{xx} = -4, \quad P_{xy} = P_{yx} = -4, \quad \text{and} \quad P_{yy} = -8.$$

Thus $D = P_{xx}P_{yy} - P_{xy}^2 = (-4)(-8) - (-4)^2 = 16 > 0$ and $P_{xx} < 0$, so this is indeed a local maximum.

- 3 Find the normal system that you would use to solve for the line $y = mx + b$ that is the least squares best fit to the data

$$(1, 1) \quad (3, 4) \quad (5, 6) \quad (7, 6).$$

You do not need to solve for m and b ! Just set up the system!

Solution: One way to set this up is to write down the linear system as $A^TAX = A^TY$, where

$$A = \begin{bmatrix} 1 & 1 \\ 3 & 1 \\ 5 & 1 \\ 7 & 1 \end{bmatrix}, \quad X = \begin{bmatrix} m \\ b \end{bmatrix}, \quad \text{and} \quad Y = \begin{bmatrix} 1 \\ 4 \\ 6 \\ 6 \end{bmatrix}.$$

Since

$$A^T A = \begin{bmatrix} 1 & 3 & 5 & 7 \\ 1 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 3 & 1 \\ 5 & 1 \\ 7 & 1 \end{bmatrix} = \begin{bmatrix} 1^2 + 3^2 + 5^2 + 7^2 & 1 + 3 + 5 + 7 \\ 1 + 3 + 5 + 7 & 1 + 1 + 1 + 1 \end{bmatrix} = \begin{bmatrix} 84 & 16 \\ 16 & 4 \end{bmatrix}$$

and

$$A^T Y = \begin{bmatrix} 1 & 3 & 5 & 7 \\ 1 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 4 \\ 6 \\ 6 \end{bmatrix} = \begin{bmatrix} 1(1) + 3(4) + 5(6) + 7(6) \\ 1(1) + 1(4) + 1(6) + 1(6) \end{bmatrix} = \begin{bmatrix} 85 \\ 17 \end{bmatrix},$$

this normal system is simply

$$\begin{bmatrix} 84 & 16 \\ 16 & 4 \end{bmatrix} \begin{bmatrix} m \\ b \end{bmatrix} = \begin{bmatrix} 85 \\ 17 \end{bmatrix}. \quad (*)$$

This is our answer.

Another approach is to write down the normal system as

$$\begin{bmatrix} \sum_{i=1}^n x_i^2 & \sum_{i=1}^n x_i \\ \sum_{i=1}^n x_i & n \end{bmatrix} \begin{bmatrix} m \\ b \end{bmatrix} = \begin{bmatrix} \sum_{i=1}^n x_i y_i \\ \sum_{i=1}^n y_i \end{bmatrix}.$$

Then we can calculate all these sums simply by writing down a table:

i	x_i	y_i	x_i^2	$x_i y_i$
1	1	1	1	1
2	3	4	9	12
3	5	6	25	30
4	7	6	49	42
Sum	16	17	84	85

From this we also get the normal system (*).