Quiz Ten Solutions

1 Suppose $f(x,y) = x^3 + 2y^2 - 6xy - 14$.

(a) Find all the critical points of the function $f(x,y)$.

Solution: The critical points are the points (x,y) where both $f_x = 0$ and $f_y = 0$. We calculate and find that

$$f_x = 3x^2 - 6y \qquad \text{and} \qquad f_y = 4y - 6x.$$

From $f_y = 0$ we see that $4y - 6x = 0$ or $y = \frac{3}{2}x$. Plugging this into $f_x = 0$, we get

$$0 = 3x^2 - 6y = 3x^2 - 6 \cdot \frac{3}{2}x = 3x^2 - 9x = 3x(x-3).$$

Thus $x = 0$ or $x = 3$. We can find the y-coordinates of our critical points by simply plugging in these numbers into $y = \frac{3}{2}x$. Thus our critical points are $(x,y) = (0,0)$ and $(x,y) = (3, 9/2)$.

(b) Classify each of the critical points you found in part (b) as a local maximum, a local minimum, or a saddle point. Recall that the tests are: at the critical point,

- if $D > 0$ and $f_{xx} < 0$, then the critical point is a local maximum;
- if $D > 0$ and $f_{xx} > 0$, then the critical point is a local minimum;
- if $D < 0$, then the critical point is a saddle point.

Solution: The second derivatives for our function $f(x,y)$ are

$$f_{xx} = 6x, \qquad f_{xy} = f_{yx} = -6, \qquad \text{and} \qquad f_{yy} = 4.$$

Thus $D = f_{xx}f_{yy} - f_{xy}^2$ is $D = (6x)(4) - (-6)^2 = 24x - 36$. At the point $(x,y) = (0,0)$, we therefore have $D = 24(0) - 36 = -36 < 0$, so $(0,0)$ is a saddle point (by the test given above). At the other critical point $(x,y) = (3, 9/2)$, the value of D is $D = 24(3) - 36 = 36 > 0$, so this critical point is either a maximum or a minimum, depending on the sign of f_{xx} at the critical point. Since $f_{xx} = 6x = 6 \cdot 3 > 0$, this critical point is a local minimum.

2 A company is making two products. The demand functions are given by

$$p_1 = 24 - 2x \qquad \text{and} \qquad p_2 = 20 - y$$

where x and y are given in thousands of units and the prices p_1 and p_2 are in thousands of dollars. The joint cost function is given by $C(x, y) = 2x^2 + 4xy + y^2$. Find the maximum possible profit.

Hint: Recall that profit is revenue minus cost: $P = R - C$.

Solution: Since we're given the cost function, we need only find the revenue function on our way to finding the profit. The revenue from selling the first product is the x, number of units sold, times p_1, the price of each unit. Similarly, the revenue from the second product is y times p_2, so the revenue function is

$$R(x, y) = xp_1 + yp_2 = x\,(24 - 2x) + y\,(20 - y) = 24x - 2x^2 + 20y - y^2.$$

Thus the profit function that we wish to maximize is

$$\begin{aligned} P(x, y) &= R(x, y) - C(x, y) \\ &= \left(24x - 2x^2 + 20y - y^2\right) - \left(2x^2 + 4xy + y^2\right) \\ &= 24x - 4x^2 - 4xy + 20y - 2y^2. \end{aligned}$$

We maximize this function in the usual way, by setting the first partial derivatives equal to zero:

$$P_x = 24 - 8x - 4y = 0 \qquad \text{and} \qquad P_y = -4x + 20 - 4y = 0.$$

This linear system simplifies to

$$\begin{aligned} 8x + 4y &= 24 \\ 4x + 4y &= 20 \end{aligned}$$

which has solution (critical point) $(x, y) = (1, 4)$. At this point the profit is

$$P(1, 4) = 24(1) - 4(1)^2 - 4(1)(4) + 20(4) - 2(4)^2 = 52,$$

or $\$52,000$.

We can check that this is a maximum (so that we haven't just minimized our profits!) by examining the second partial derivatives:

$$P_{xx} = -8, \qquad P_{xy} = P_{yx} = -4, \qquad \text{and} \qquad P_{yy} = -4.$$

Thus $D = P_{xx}P_{yy} - P_{xy}^2 = (-8)(-4) - (-4)^2 = 16 > 0$ and $P_{xx} < 0$, so this is indeed a local maximum.

3 Find the normal system that you would use to solve for the line $y = mx + b$ that is the least squares best fit to the data

$$(0, 2) \qquad (2, 5) \qquad (4, 6) \qquad (6, 7).$$

You do not need to solve for m and b! Just set up the system!

Solution: One way to set this up is to write down the linear system as $A^T AX = A^T Y$, where

$$A = \begin{bmatrix} 0 & 1 \\ 2 & 1 \\ 4 & 1 \\ 6 & 1 \end{bmatrix}, \qquad X = \begin{bmatrix} m \\ b \end{bmatrix}, \qquad \text{and} \qquad Y = \begin{bmatrix} 2 \\ 5 \\ 6 \\ 7 \end{bmatrix}.$$

Since

$$A^T A = \begin{bmatrix} 0 & 2 & 4 & 6 \\ 1 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 2 & 1 \\ 4 & 1 \\ 6 & 1 \end{bmatrix} = \begin{bmatrix} 0^2 + 2^2 + 4^2 + 6^2 & 0 + 2 + 4 + 6 \\ 0 + 2 + 4 + 6 & 1 + 1 + 1 + 1 \end{bmatrix} = \begin{bmatrix} 56 & 12 \\ 12 & 4 \end{bmatrix}$$

and

$$A^T Y = \begin{bmatrix} 0 & 2 & 4 & 6 \\ 1 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ 5 \\ 6 \\ 7 \end{bmatrix} = \begin{bmatrix} 0(2) + 2(5) + 4(6) + 6(7) \\ 1(2) + 1(5) + 1(6) + 1(7) \end{bmatrix} = \begin{bmatrix} 76 \\ 20 \end{bmatrix},$$

this normal system is simply

$$\begin{bmatrix} 56 & 12 \\ 12 & 4 \end{bmatrix} \begin{bmatrix} m \\ b \end{bmatrix} = \begin{bmatrix} 76 \\ 20 \end{bmatrix}. \qquad (*)$$

This is our answer.

Another approach is to write down the normal system as

$$\begin{bmatrix} \sum_{i=1}^{n} x_i^2 & \sum_{i=1}^{n} x_i \\ \sum_{i=1}^{n} x_i & n \end{bmatrix} \begin{bmatrix} m \\ b \end{bmatrix} = \begin{bmatrix} \sum_{i=1}^{n} x_i y_i \\ \sum_{i=1}^{n} y_i \end{bmatrix}.$$

Then we can calculate all these sums simply by writing down a table:

i	x_i	y_i	x_i^2	$x_i y_i$
1	0	2	0	0
2	2	5	4	10
3	4	6	16	24
4	6	7	36	42
Sum	12	20	56	76

From this we also get the normal system $(*)$.