

1 The equation

$$x^2 + y^2 + z^2 + 2x - 6y + z = 2$$

describes a sphere. Find the center and the radius of this sphere.

**Solution:** The key to this is the technique of completing the square. That is, we'd like to write this equation as

$$(x - a)^2 + (y - b)^2 + (z - c)^2 = r^2,$$

as we could then say that the center is  $(x, y, z) = (a, b, c)$  and the radius is  $r$ . If we expand this out we get

$$(x^2 - 2ax + a^2) + (y^2 - 2by + b^2) + (z^2 - 2cz + c^2) = r^2.$$

Let us re-write our original equation to fit this model:

$$(x^2 + 2x + \underline{\quad}) + (y^2 - 6y + \underline{\quad}) + (z^2 + z + \underline{\quad}) = 2$$

From this we see that  $-2a = 2$ , so  $a = -1$ . Similarly  $-2b = -6$ , so  $b = 3$  and  $-2c = 1$ , so  $c = 1/2$ . Thus, adding to both sides to preserve equality, we get

$$(x^2 + 2x + (-1)^2) + (y^2 - 6y + (3)^2) + (z^2 + z + (1/2)^2) = 2 + (-1)^2 + (3)^2 + (1/2)^2$$

Thus our equation is

$$(x + 1)^2 + (y - 3)^2 + (z + 1/2)^2 = 12.25 = 3.5^2$$

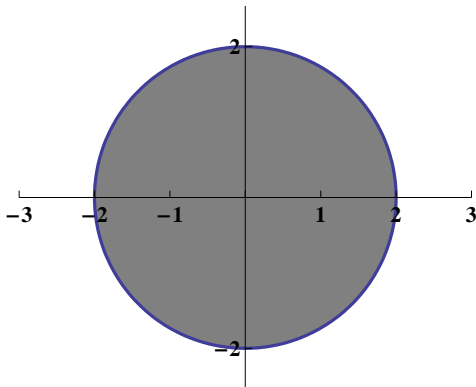
That is, our circle has center  $(a, b, c) = (-1, 3, -1/2)$  and radius  $r = 3.5$ .

2 Suppose  $f(x, y) = \sqrt{4 - x^2 - y^2}$ . Describe the domain of  $f(x, y)$ . Graph this domain on the axes provided.

**Solution:** The only issue here is that the argument of the square root must not be negative. That is, we must have

$$4 - x^2 - y^2 \geq 0 \quad \text{or} \quad x^2 + y^2 \leq 4.$$

With equality, this is a circle  $x^2 + y^2 = 2^2$ , centered at the origin with radius 2. With the inequality, this is everything *inside* this circle. Since it is a non-strict inequality (that is, a less-than-or-equal-to), we also include the boundary circle. Here is a picture, with the domain shaded:



- 3 Suppose  $f(x, y) = e^{2xy} + 4x^2y - 3x^2 + xy + 2$ . Calculate all the first and second partial derivatives:  $f_x$ ,  $f_y$ ,  $f_{xx}$ ,  $f_{xy}$ ,  $f_{yx}$ , and  $f_{yy}$ .

**Solution:** The computations here are straightforward, so we give only the answers.

First derivatives:

$$f_x = 2ye^{2xy} + 8xy - 6x + y \quad \text{and} \quad f_y = 2xe^{2xy} + 4x^2 + x.$$

Second derivatives:

$$f_{xx} = 4y^2e^{2xy} + 8y - 6 \quad \text{and} \quad f_{yy} = 4x^2e^{2xy}$$

and

$$f_{xy} = f_{yx} = 2e^{2xy} + 4xye^{2xy} + 8x + 1 = (2 + 4xy)e^{2xy} + 8x + 1.$$

Notice that  $f_{xy} = f_{yx}$ , as usual.

- 4 BOB the writer of textbook answers tells you that for some function  $g(x, y)$ , the first partial derivatives are  $g_x = 2x + 5y$  and  $g_y = 4x - 3y$ . Explain why BOB must be wrong. Use some (or all) of the second partial derivatives to justify your answer.

**Solution:** The key here is that  $g_{xy}$  must equal  $g_{yx}$ . Even though we don't know  $g(x, y)$ , we can still compute  $g_{xy}$  from  $g_x$  and  $g_{yx}$  from  $g_y$ . When we do these computations, we get

$$g_{xy} = \frac{\partial}{\partial y}(g_x) = \frac{\partial}{\partial y}(2x + 5y) = 0 + 5 \cdot 1 = 5$$

and

$$g_{yx} = \frac{\partial}{\partial x}(g_y) = \frac{\partial}{\partial x}(4x - 3y) = 4 \cdot 1 - 0 = 4.$$

Since these aren't equal, Bob must have made a mistake in computing  $g_x$  or  $g_y$  (or both).

Another approach would be to try to figure out what  $g(x, y)$  was originally. If we start with  $g_x = 2x + 5y$  and integrate with respect to  $x$ , we'd get something like

$$g(x, y) = x^2 + 5xy + K.$$

The trick here is that this constant  $K$  is only constant *with respect to*  $x$ , and therefore it is really a function of  $y$ . That is,

$$g(x, y) = x^2 + 5xy + K(y),$$

so if we were to compute  $g_y$ , we'd get  $g_y = 0 + 5x + K'(y)$ . Now we'd have to argue that  $5x + K'(y)$  can't equal  $4x - 3y$  (Bob's answer), since this would mean  $K'(y) = -x - 3y$ . This won't work, since  $K(y)$  (and thus also  $K'(y)$ ) is a function of only  $y$ .

People do use this approach (talk to your friends in Math 223), but we haven't and won't. This certainly wasn't what we expected you to do.