Quiz Nine Solutions

1 The equation

$$x^2 + y^2 + z^2 - 4x + 2y - z = 1$$

describes a sphere. Find the center and the radius of this sphere.

Solution: The key to this is the technique of completing the square. That is, we'd like to write this equation as

$$(x-a)^2 + (y-b)^2 + (z-c)^2 = r^2,$$

as we could then say that the center is $(x, y, z) = (a, b, c)$ and the radius is r. If we expand this out we get

$$\left(x^2 - 2ax + a^2\right) + \left(y^2 - 2by + b^2\right) + \left(z^2 - 2cz + c^2\right) = r^2.$$

Let us re-write our original equation to fit this model:

$$\left(x^2 - 4x + \_\_\ \right) + \left(y^2 + 2y + \_\_\ \right) + \left(z^2 - z + \_\_\ \right) = 1$$

From this we see that $-2a = -4$, so $a = 2$. Similarly $-2b = 2$, so $b = -1$ and $-2c = -1$, so $c = {}^{-1}\!/_2$. Thus, adding to both sides to preserve equality, we get

$$\left(x^2 - 4x + (2)^2\right) + \left(y^2 + 2y + (-1)^2\right) + \left(z^2 - z + \left({}^{-1}\!/_2\right)^2\right) = 1 + (2)^2 + (-1)^2 + \left({}^{-1}\!/_2\right)^2$$

Thus our equation is

$$(x-2)^2 + (y+1)^2 + \left(z - {}^{1}\!/_2\right)^2 = 6.25 = 2.5^2.$$

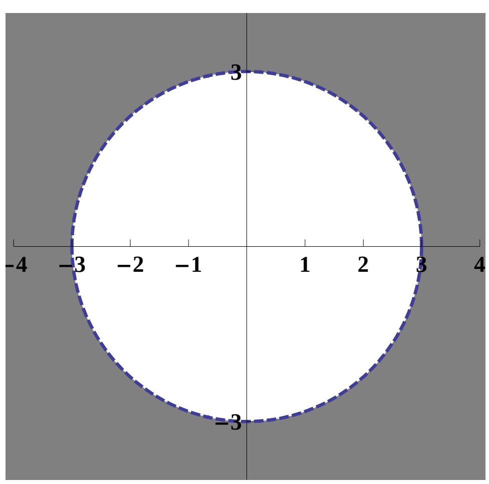
That is, our circle has center $(a, b, c) = (2, -1, {}^{1}\!/_2)$ and radius $r = 2.5$.

2 Suppose $f(x, y) = \ln\left(2x^2 + 2y^2 - 18\right)$. Describe the domain of $f(x, y)$. Graph this domain on the axes provided.

Solution: The only issue here is that the argument of the natural logarithm must be positive. That is, we must have

$$2x^2 + 2y^2 - 18 > 0 \qquad \text{or} \qquad x^2 + y^2 > 9.$$

With equality, this is a circle $x^2 + y^2 = 3^2$, centered at the origin with radius 3. With the inequality, this is everything *outside* this circle. Since it is a strict inequality, we do not include the boundary circle. Here is a picture, with the domain shaded:



3 Suppose $f(x, y) = e^{3xy} + 3x^2y - 2y^2 + 15$. Calculate all the first and second partial derivatives: f_x, f_y, f_{xx}, f_{xy}, f_{yx}, and f_{yy}.

Solution: The computations here are straightforward, so we give only the answers.

First derivatives:

$$f_x = 3ye^{3xy} + 6xy \qquad \text{and} \qquad f_y = 3xe^{3xy} + 3x^2 - 4y.$$

Second derivatives:

$$f_{xx} = 9y^2e^{3xy} + 6y \qquad \text{and} \qquad f_{yy} = 9x^2e^{3xy} - 4$$

and

$$f_{xy} = f_{yx} = 3e^{3xy} + 9xye^{3xy} + 6x = (3 + 9xy)e^{3xy} + 6x.$$

Notice that $f_{xy} = f_{yx}$, as usual.

4 BOB the writer of textbook answers tells you that for some function $g(x, y)$, the first partial derivatives are $g_x = 5x - 4y$ and $g_y = 3x + 4y$. Explain why BOB must be wrong. Use some (or all) of the second partial derivatives to justify your answer.

Solution: The key here is that g_{xy} must equal g_{yx}. Even though we don't know $g(x, y)$, we can still compute g_{xy} from g_x and g_{yx} from g_y. When we do these computations, we get

$$g_{xy} = \frac{\partial}{\partial y}\big(g_x\big) = \frac{\partial}{\partial y}\Big(5x - 4y\Big) = 0 - 4 \cdot 1 = -4$$

and

$$g_{yx} = \frac{\partial}{\partial x}\big(g_y\big) = \frac{\partial}{\partial x}\Big(3x + 4y\Big) = 3 \cdot 1 + 0 = 3.$$

Since these aren't equal, Bob must have made a mistake in computing g_x or g_y (or both).

Another approach would be to try to figure out what $g(x, y)$ was originally. If we start with $g_x = 5x - 4y$ and integrate with respect to x, we'd get something like

$$g(x, y) = \frac{5}{2}x^2 - 4xy + K.$$

The trick here is that this constant K is only constant *with respect to* x, and therefore it is really a function of y. That is,

$$g(x, y) = \frac{5}{2}x^2 - 4xy + K(y),$$

so if we were to compute g_y, we'd get $g_y = 0 - 4x + K'(y)$. Now we'd have to argue that $-4x + K'(y)$ can't equal $3x + 4y$ (Bob's answer), since this would mean $K'(y) = 7x + 4y$. This won't work, since $K(y)$ (and thus also $K'(y)$) is a function of only y.

People do use this approach (talk to your friends in Math 223), but we haven't and won't. This certainly wasn't what we expected you to do.