Quiz Eight Solutions

1 Suppose $AX = B$, where $B = \begin{bmatrix} 1 \\ 2 \\ -2 \end{bmatrix}$ and $A^{-1} = \begin{bmatrix} 2 & -1 & 1 \\ 0 & -2 & 3 \\ 1 & 0 & 2 \end{bmatrix}$. Find X.

Solution: Since $AX = B$, we may find X by multiplying by A^{-1} to get $X = A^{-1}B$:

$$A^{-1}B = A^{-1}AX = IX = X.$$

Thus we get

$$X = A^{-1}B = \begin{bmatrix} 2 & -1 & 1 \\ 0 & -2 & 3 \\ 1 & 0 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ -2 \end{bmatrix} = \begin{bmatrix} 2(1) - 1(2) + 1(-2) \\ 0(1) - 2(2) + 3(-2) \\ 1(1) + 0(2) + 2(-2) \end{bmatrix} = \begin{bmatrix} -2 \\ -10 \\ -3 \end{bmatrix}.$$

This is our answer.

2 The function $y = 3x^2 + ax$ is a solution of the differential equation

$$xy' = 2y + 2x.$$

What is the constant a? Verify that your function (with your a) is a solution.

Solution: The easiest way to find a is to simply plug in $y = 3x^2 + ax$ (and $y' = 6x + a$) into the differential equation. We do this and get

$$x\,(6x + a) = 2\left(3x^2 + ax\right) + 2x$$

or

$$6x^2 + ax = 6x^2 + 2ax + 2x$$

This simplifies to $ax = (2a+2)x$, so $a = -2$. Thus $y = 3x^2 - 2x$ is a solution to this differential equation:

$$xy' = x(6x-2) = 6x^2 - 2x \qquad \text{and} \qquad 2y + 2x = 2(3x^2 - 2x) + 2x = 6x^2 - 4x + 2x = 6x^2 - 2x.$$

Since these agree, the given function is a solution.

3 Find the general solution to the differential equation

$$e^{2y}y' = x^4 + 7.$$

Solution: This equation is essentially separated for us; all we need to do is replace y' with $\frac{dy}{dx}$ and multiply through by dx:

$$e^{2y}\,dy = \left(x^4 + 7\right)\,dx.$$

Now we integrate both sides to get

$$\int e^{2y}\,dy = \int \left(x^4 + 7\right)\,dx$$

or

$$\frac{1}{2}e^{2y} = \frac{1}{5}x^5 + 7x + K.$$

To solve for y, we first multiply by 2:

$$e^{2y} = \frac{2}{5}x^5 + 14x + 2K = \frac{2}{5}x^5 + 14x + C$$

(where $C = 2K$ is an unknown constant), then take the natural logarithm:

$$2y = \ln\left(e^{2y}\right) = \ln\left(\frac{2}{5}x^5 + 14x + C\right).$$

Finally, we divide both sides by 2 to get our final answer:

$$y = \frac{1}{2}\ln\left(\frac{2}{5}x^5 + 14x + C\right).$$

4 Find the solution of $y' = \frac{x^4}{y^4} - \frac{3}{x^4y^4}$ through the point $(x, y) = (1, 1)$.

Solution: This equation is separable, but (unlike the previous question) we need to actually separate it. Let's factor the y^4 out of the denominator on the right-hand side and re-write the equation as

$$\frac{dy}{dx} = \frac{1}{y^4}\left(x^4 - \frac{3}{x^4}\right) = \frac{1}{y^4}\left(x^4 - 3x^{-4}\right).$$

Now we multiply by $y^4\ dx$ to get

$$y^4\ dy = \left(x^4 - 3x^{-4}\right)\ dx.$$

Now integrating both sides gives us

$$\int y^4\ dy = \int \left(x^4 - 3x^{-4}\right)\ dx$$

or

$$\frac{1}{5}y^5 = \frac{1}{5}x^5 - \frac{3}{-3}x^{-3} + K \qquad \text{or} \qquad \frac{1}{5}y^5 = \frac{1}{5}x^5 + \frac{1}{x^3} + K.$$

We multiply this equation by 4, then take the fifth root to get

$$y = \sqrt[5]{x^5 + \frac{5}{x^3} + C}$$

(where $C = 4K$ is an unknown constant). Plugging in the point $(x, y) = (1, 1)$, we get

$$1 = \sqrt[5]{1^5 + \frac{5}{1^3} + C} = \sqrt[5]{6 + C}$$

so $C = -5$. Thus our final answer is

$$y = \sqrt[5]{x^5 + \frac{5}{x^3} - 5} \qquad \text{or} \qquad y = \left(x^5 + \frac{5}{x^3} - 5\right)^{1/5}.$$