

1 Suppose $AX = B$, where $B = \begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix}$ and $A^{-1} = \begin{bmatrix} 1 & 0 & 2 \\ 2 & -1 & 1 \\ 0 & -2 & 3 \end{bmatrix}$. Find X .

Solution: Since $AX = B$, we may find X by multiplying by A^{-1} to get $X = A^{-1}B$:

$$A^{-1}B = A^{-1}AX = IX = X.$$

Thus we get

$$X = A^{-1}B = \begin{bmatrix} 1 & 0 & 2 \\ 2 & -1 & 1 \\ 0 & -2 & 3 \end{bmatrix} \begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix} = \begin{bmatrix} 1(2) + 0(-2) + 2(1) \\ 2(2) - 1(-2) + 1(1) \\ 0(2) - 2(-2) + 3(1) \end{bmatrix} = \begin{bmatrix} 4 \\ 7 \\ 7 \end{bmatrix}.$$

This is our answer.

2 The function $y = 4x^2 + ax$ is a solution of the differential equation

$$xy' = 2y + x.$$

What is the constant a ? Verify that your function (with your a) is a solution.

Solution: The easiest way to find a is to simply plug in $y = 4x^2 + ax$ (and $y' = 8x + a$) into the differential equation. We do this and get

$$x(8x + a) = 2(4x^2 + ax) + x(6x + a) = 2(3x^2 + ax) + 2x$$

or

$$8x^2 + ax = 8x^2 + 2ax + x$$

This simplifies to $ax = (2a + 1)x$, so $a = -1$. Thus $y = 4x^2 - x$ is a solution to this differential equation:

$$xy' = x(8x - 1) = 8x^2 - x \quad \text{and} \quad 2y + x = 2(4x^2 - x) + x = 8x^2 - 2x + x = 8x^2 - x.$$

Since these agree, the given function is a solution.

3 Find the general solution to the differential equation

$$e^{3y}y' = x^3 - 5.$$

Solution: This equation is essentially separated for us; all we need to do is replace y' with $\frac{dy}{dx}$ and multiply through by dx :

$$e^{3y} dy = (x^3 - 5) dx.$$

Now we integrate both sides to get

$$\int e^{3y} dy = \int (x^3 - 5) dx$$

or

$$\frac{1}{3}e^{3y} = \frac{1}{4}x^4 - 5x + K.$$

To solve for y , we first multiply by 3:

$$e^{3y} = \frac{3}{4}x^4 - 15x + 3K = \frac{3}{4}x^4 - 15x + C$$

(where $C = 3K$ is an unknown constant), then take the natural logarithm:

$$3y = \ln(e^{3y}) = \ln\left(\frac{3}{4}x^4 - 15x + C\right).$$

Finally, we divide both sides by 3 to get our final answer:

$$y = \frac{1}{3} \ln\left(\frac{3}{4}x^4 - 15x + C\right).$$

4 Find the solution of $y' = \frac{x^2}{y^2} - \frac{2}{x^2y^2}$ through the point $(x, y) = (1, 1)$.

Solution: This equation is separable, but (unlike the previous question) we need to actually separate it. Let's factor the y^2 out of the denominator on the right-hand side and re-write the equation as

$$\frac{dy}{dx} = \frac{1}{y^2} \left(x^2 - \frac{2}{x^2}\right) = \frac{1}{y^2} (x^2 - 2x^{-2}).$$

Now we multiply by $y^2 dx$ to get

$$y^2 dy = (x^2 - 2x^{-2}) dx.$$

Now integrating both sides gives us

$$\int y^2 dy = \int (x^2 - 2x^{-2}) dx$$

or

$$\frac{1}{3}y^3 = \frac{1}{3}x^3 - \frac{2}{-1}x^{-1} + K \quad \text{or} \quad \frac{1}{3}y^3 = \frac{1}{3}x^3 + \frac{2}{x} + K.$$

We multiply this equation by 3, then take the cube root to get

$$y = \sqrt[3]{x^3 + \frac{6}{x} + C}$$

(where $C = 3K$ is an unknown constant). Plugging in the point $(x, y) = (1, 1)$, we get

$$1 = \sqrt[3]{1^3 + \frac{6}{1} + C} = \sqrt[3]{7 + C}$$

so $C = -6$. Thus our final answer is

$$y = \sqrt[3]{x^3 + \frac{6}{x} - 6} \quad \text{or} \quad y = \left(x^3 + \frac{6}{x} - 6\right)^{1/3}.$$