Quiz Seven Solutions

1 Consider the matrix $A = \begin{bmatrix} 3 & 2 & 1 \\ -2 & & \\ 2 & & \end{bmatrix}$. Something is wrong with our printer, because we're missing some of the entries. We're told that one of the following matrices (which our printer has also misprinted) is actually the inverse matrix A^{-1}. Decide which matrix is actually A^{-1}. Justify your answer with some computations.

$$B = \begin{bmatrix} 1 & 2 & 1 \\ 2 & & \\ -6 & & \end{bmatrix} \qquad C = \begin{bmatrix} 2 & 0 & 1 \\ 6 & & \\ -1 & & \end{bmatrix} \qquad D = \begin{bmatrix} 1 & 1 & -1 \\ 2 & & \\ -6 & & \end{bmatrix}$$

Solution: If our printer wasn't broken, we could do this by simply computing A^{-1}, then pick the answer from the list. This is much more work, however, than we really need to do.

The main property of the inverse A^{-1} is that $AA^{-1} = I$ and $A^{-1}A = I$. Because of the missing entries in our matrices, we can only compute the $(1,1)$ entry (the entry in row 1 and column 1) of the products AB, AC, and AD, as well as BA, CA, and DA. We do this and find that B is the only possible inverse for A.

Here are the computations:

$$AB_{(1,1)} = \begin{bmatrix} 3 & 2 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ -6 \end{bmatrix} = 3 \cdot 1 + 2 \cdot 2 + 1 \cdot -6 = 3 + 4 - 6 = 1,$$

$$AC_{(1,1)} = \begin{bmatrix} 3 & 2 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ 6 \\ -1 \end{bmatrix} = 3 \cdot 2 + 2 \cdot 6 + 1 \cdot -1 = 6 + 12 - 1 = 17,$$

$$AD_{(1,1)} = \begin{bmatrix} 3 & 2 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ -6 \end{bmatrix} = 3 \cdot 1 + 2 \cdot 2 + 1 \cdot -6 = 3 + 4 - 6 = 1.$$

Similarly,

$$BA_{(1,1)} = \begin{bmatrix} 1 & 2 & 1 \end{bmatrix} \begin{bmatrix} 3 \\ -2 \\ 2 \end{bmatrix} = 1 \cdot 3 + 2 \cdot -2 + 1 \cdot 2 = 3 - 4 + 2 = 1,$$

$$CA_{(1,1)} = \begin{bmatrix} 2 & 0 & 1 \end{bmatrix} \begin{bmatrix} 3 \\ -2 \\ 2 \end{bmatrix} = 2 \cdot 3 + 0 \cdot -2 + 1 \cdot 2 = 6 + 0 + 2 = 8,$$

$$DA_{(1,1)} = \begin{bmatrix} 1 & 1 & -1 \end{bmatrix} \begin{bmatrix} 3 \\ -2 \\ 2 \end{bmatrix} = 1 \cdot 3 + 1 \cdot -2 + -1 \cdot 2 = 3 - 2 - 2 = -1.$$

Thus the only one with a 1 in the $(1,1)$ position of both multiplications is B. Thus B must be the inverse.

In particular, we can see that C and D are *not* the inverse of A. For example, C is not A^{-1} because

$$AC = \begin{bmatrix} 3 & 2 & 1 \\ -2 & * & * \\ 2 & * & * \end{bmatrix} \begin{bmatrix} 2 & 0 & 1 \\ 6 & * & * \\ -1 & * & * \end{bmatrix} = \begin{bmatrix} 17 & * & * \\ * & * & * \\ * & * & * \end{bmatrix} \neq \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

where $*$ symbolizes an entry we don't know. Similarly,

$$DA = \begin{bmatrix} 1 & 1 & -1 \\ 2 & * & * \\ -6 & * & * \end{bmatrix} \begin{bmatrix} 3 & 2 & 1 \\ -2 & * & * \\ 2 & * & * \end{bmatrix} = \begin{bmatrix} -1 & * & * \\ * & * & * \\ * & * & * \end{bmatrix} \neq \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Since the entry in the first row and first column of AA^{-1} (and $A^{-1}A$) must be 1, neither C nor D can be A^{-1}.

2 For what values of c does the matrix $A = \begin{bmatrix} 2 & 6 \\ 5 & c \end{bmatrix}$ have an inverse? Explain your reasoning.

Solution: This problem is very simple if we remember that the inverse of $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is $\frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$ whenever $ad - bc \neq 0$. This matrix has no inverse when $ad - bc = 0$. In our case, this means A has an inverse precisely when $2(c) - 6(5) \neq 0$, or $c \neq 15$.

Even if we don't remember the formula for the inverse of a 2×2 matrix, we can still compute the inverse using elimination (or row reduction). Let's see when this works:

$$\begin{aligned}
[\,A\,|\,I\,] &= \left[\begin{array}{cc|cc} 2 & 6 & 1 & 0 \\ 5 & c & 0 & 1 \end{array}\right] \\
&\longrightarrow \left[\begin{array}{cc|cc} 1 & 3 & 0.5 & 0 \\ 5 & c & 0 & 1 \end{array}\right] \qquad R_1 = \tfrac{1}{2}r_1 \\
&\longrightarrow \left[\begin{array}{cc|cc} 1 & 3 & 0.5 & 0 \\ 0 & c-15 & -2.5 & 1 \end{array}\right] \qquad R_2 = r_2 - 5r_1 \\
&\longrightarrow \left[\begin{array}{cc|cc} 1 & 3 & 0.5 & 0 \\ 0 & 1 & -\frac{2.5}{c-15} & \frac{1}{c-15} \end{array}\right] \qquad R_2 = \tfrac{1}{c-15} \cdot r_2
\end{aligned}$$

We could continue, but it should be clear that this will work (we have two pivots – one in each row, one in each column). The only thing that might have gone wrong is that we divided by $c - 15$. So as long as $c \neq 15$, we'll get an inverse. Moreover, if $c = 15$, we don't get a second pivot, so there is no inverse. Thus we get the same answer as before: the matrix A has an inverse provided $c \neq 15$.

3 Solve the following system by finding the reduced row echelon form of the associated augmented matrix. If there is no solution, say that the system is inconsistent.

$$\begin{aligned} x_1 + 2x_2 \qquad\quad - \ x_4 &= \ \ 4 \\ 2x_1 + 5x_2 + \ x_3 + \ x_4 &= \ \ 0 \\ 3x_1 + 6x_2 + 2x_3 + 5x_4 &= -4 \end{aligned}$$

Note:

- You must show the details of your computation to receive full credit!
- You must put the matrix in *reduced* row echelon form, not simply row echelon form!

Solution: We begin by turning this into an augmented matrix; it becomes

$$\left[\begin{array}{cccc|c} 1 & 2 & 0 & -1 & 4 \\ 2 & 5 & 1 & 1 & 0 \\ 3 & 6 & 2 & 5 & -4 \end{array}\right]$$

We will perform our elimination, noting the steps taken to the right:

$$\left[\begin{array}{cccc|c} 1 & 2 & 0 & -1 & 4 \\ 2 & 5 & 1 & 1 & 0 \\ 3 & 6 & 2 & 5 & -4 \end{array}\right] \longrightarrow \left[\begin{array}{cccc|c} 1 & 2 & 0 & -1 & 4 \\ 0 & 1 & 1 & 3 & -8 \\ 0 & 0 & 2 & 8 & -16 \end{array}\right] \qquad \begin{array}{l} R_2 = r_2 - 2r_1 \\ R_3 = r_3 - 3r_1 \end{array}$$

$$\longrightarrow \left[\begin{array}{cccc|c} 1 & 2 & 0 & -1 & 4 \\ 0 & 1 & 1 & 3 & -8 \\ 0 & 0 & 1 & 4 & -8 \end{array}\right] \qquad R_3 = \tfrac{1}{2} \cdot r_3$$

$$\longrightarrow \left[\begin{array}{cccc|c} 1 & 2 & 0 & -1 & 4 \\ 0 & 1 & 0 & -1 & 0 \\ 0 & 0 & 1 & 4 & -8 \end{array}\right] \qquad R_2 = r_2 - r_3$$

$$\longrightarrow \left[\begin{array}{cccc|c} 1 & 0 & 0 & 1 & 4 \\ 0 & 1 & 0 & -1 & 0 \\ 0 & 0 & 1 & 4 & -8 \end{array}\right] \qquad R_1 = r_1 - 2r_2$$

This augmented matrix (now in reduced row echelon form) corresponds to the linear system

$$\begin{aligned} x_1 \qquad\quad + \ x_4 &= \ \ 4 \\ x_2 \qquad - \ x_4 &= \ \ 0 \\ x_3 + 4x_4 &= -8 \end{aligned}$$

This system has solution

$$(x_1, x_2, x_3, x_4) = (4 - x_4, x_4, -8 - 4x_4, x_4) \qquad \text{or} \qquad \begin{cases} x_1 &= 4 - x_4 \\ x_2 &= x_4 \\ x_3 &= -8 - 4x_4 \\ x_4 &= x_4. \end{cases}$$