Quiz Five Solutions

1 The Cleveland Cavaliers' old starting line-up for last Tuesday's game featured players age 23, 23, 26, 29, and 32. Find both the mean and median age of this line-up.

Solution: The median is the middle number, which in this case is 26 years.

The mean, or average, is only slightly more complicated:

$$\text{mean} = \mu = \frac{23 + 23 + 26 + 29 + 32}{5} = 26.6 \text{ years.}$$

If we let X be the random variable that has equally likely outcomes the ages of the five starters, then we're saying that $E(X) = \mu = 26.6$ years.

2 Find the standard deviation (assuming population data) of the ages of the Cavaliers' line-up from problem 1.

Solution: Recall that the standard deviation σ is the square root of the variance $\sigma^2 = \text{Var}(X)$. One way to compute the variance is as $\text{Var}(X) = E\left((X - \mu)^2\right)$, but a simpler way is simply as $\text{Var}(X) = E(X^2) - \mu^2$. By this expression $E(X^2)$ we mean

$$E(X^2) = \frac{23^2 + 23^2 + 26^2 + 29^2 + 32^2}{5} = \frac{3599}{5} = 719.8.$$

Since $\mu = 26.6$ (from part (a)), we have

$$\sigma^2 = \text{Var}(X) = E(X^2) - \mu^2 = 719.8 - 26.6^2 = 12.24.$$

Thus the standard deviation is $\sigma = \sqrt{12.24} \approx 3.4986$ years.

3 A candy company W&W's sells small candies by the bag (each imprinted with a small W). An audit of the production process shows that there are an average of $\mu = 540$ W's in each medium-sized bag, with a variance of about $\sigma^2 = 20$. Use Chebychev's theorem to estimate the probability that a medium-sized bag of candies will have between 510 and 570 (inclusive) W's.

Solution: Recall that Chebychev's theorem says the following. If X is a random variable with mean μ and variance σ^2, then

$$\Pr(\mu - k \leq X \leq \mu + k) \geq 1 - \frac{\sigma^2}{k^2}.$$

Here $\mu = 540$ and $\sigma^2 = 20$ and we're asked to find $\Pr(510 \leq X \leq 570)$, so $540 - k = 510$ and $540 + k = 570$, which means $k = 30$. Chebychev's theorem thus says that this probability is

$$\Pr(510 \leq X \leq 570) \geq 1 - \frac{\sigma^2}{k^2} = 1 - \frac{20}{30^2} = 1 - \frac{20}{900} = \frac{44}{45} \approx 97.78\%.$$

Thus the probability that a medium-sized bag of candies will have between 510 and 570 (inclusive) W's is at least 97.78%.

4 Choose k, if possible, so that $f(x) = \frac{k}{x^3}$ is a probability density function on the interval $[1, 2]$. If this is not possible, explain why.

Solution: To be a probability density function, $f(x)$ must be non-negative on the interval $[1, 2]$ and integrate to 1 over this interval. The constant k will be determined by this second condition, after which we can check that the first condition is also satisfied.

We integrate $f(x)$ from $x = 1$ to $x = 2$ in order to determine k:

$$1 = \int_1^2 f(x)\,dx = \int_1^2 \frac{k}{x^3}\,dx = k\int_1^2 x^{-3}\,dx = k\left(\frac{1}{-2}x^{-2}\right)\Big|_1^2.$$

We evaluate this at the endpoints to get

$$1 = k\left(-\frac{1}{2x^2}\right)\Big|_1^2 = k\left[-\frac{1}{2\cdot 2^2} - \left(-\frac{1}{2\cdot 1^2}\right)\right] = k\left(-\frac{1}{8}+\frac{1}{2}\right) = \frac{3}{8}\cdot k.$$

Thus $k = \frac{8}{3} \approx 2.6667$.

Notice that k is positive, so $f(x)$ is positive on the interval $[1, 2]$ as well. Thus the first condition we listed is automatically satisfied by $f(x)$ for this choice of k.

5 A number x is selected at random from the interval $[0, 4]$. The probability density function for x is

$$f(x) = \frac{1}{2} - \frac{1}{8}x \qquad \text{for } 0 \le x \le 4.$$

Find the probability that a number is selected in the subinterval $[0, 2]$.

Solution: If we let X be the random variable that is described by this problem, then the question asks for $\Pr(0 \le X \le 2)$. Since X is a continuous random variable with probability density function $f(x)$, this probability is

$$\Pr(0 \le X \le 2) = \int_0^2 f(x)\,dx = \int_0^2 \left(\frac{1}{2} - \frac{1}{8}x\right)\,dx.$$

This is just a computation:

$$\begin{aligned}\Pr(0 \le X \le 2) &= \int_0^2 \left(\frac{1}{2} - \frac{1}{8}x\right)\,dx = \left(\frac{1}{2}x - \frac{1}{8}\cdot\frac{1}{2}x^2\right)\Big|_0^2 \\ &= \left(\frac{1}{2}\cdot 2 - \frac{1}{16}\cdot 2^2\right) - \left(\frac{1}{2}\cdot 0 - \frac{1}{16}\cdot 0^2\right) = 1 - \frac{1}{4} = \frac{3}{4}.\end{aligned}$$

Thus the probability that the selected number is in the given subinterval is $\frac{3}{4}$ or 75%.