Quiz Five Solutions

1 The Cleveland Cavaliers' new starting line-up for Sunday's game featured players age 23, 24, 29, 32, and 33. Find both the mean and median age of this line-up.

Solution: The median is the middle number, which in this case is 29 years.

The mean, or average, is only slightly more complicated:

$$\text{mean} = \mu = \frac{23+24+29+32+33}{5} = 28.2 \text{ years.}$$

If we let X be the random variable that has equally likely outcomes the ages of the five starters, then we're saying that $E(X) = \mu = 28.2$ years.

2 Find the standard deviation (assuming population data) of the ages of the Cavaliers' line-up from problem 1.

Solution: Recall that the standard deviation σ is the square root of the variance $\sigma^2 = \mathrm{Var}(X)$. One way to compute the variance is as $\mathrm{Var}(X) = E\left((X-\mu)^2\right)$, but a simpler way is simply as $\mathrm{Var}(X) = E(X^2) - \mu^2$. By this expression $E(X^2)$ we mean

$$E(X^2) = \frac{23^2+24^2+29^2+32^2+33^2}{5} = \frac{4059}{5} = 811.8.$$

Since $\mu = 28.2$ (from part (a)), we have

$$\sigma^2 = \mathrm{Var}(X) = E(X^2) - \mu^2 = 811.8 - 28.2^2 = 16.56.$$

Thus the standard deviation is $\sigma = \sqrt{16.56} \approx 4.0694$ years.

3 A candy company W&W's sells small candies by the bag (each imprinted with a small W). An audit of the production process shows that there are an average of $\mu = 450$ W's in each medium-sized bag, with a variance of about $\sigma^2 = 30$. Use Chebychev's theorem to estimate the probability that a medium-sized bag of candies will have between 430 and 470 (inclusive) W's.

Solution: Recall that Chebychev's theorem says the following. If X is a random variable with mean μ and variance σ^2, then

$$\Pr(\mu - k \le X \le \mu + k) \ge 1 - \frac{\sigma^2}{k^2}.$$

Here $\mu = 450$ and $\sigma^2 = 30$ and we're asked to find $\Pr(430 \le X \le 470)$, so $450 - k = 430$ and $450 + k = 470$, which means $k = 20$. Chebychev's theorem thus says that this probability is

$$\Pr(430 \le X \le 470) \ge 1 - \frac{\sigma^2}{k^2} = 1 - \frac{30}{20^2} = 1 - \frac{30}{400} = \frac{37}{40} = 92.50\%.$$

Thus the probability that a medium-sized bag of candies will have between 430 and 470 (inclusive) W's is at least 92.50%.

4 Choose k, if possible, so that $f(x) = \dfrac{k}{x^4}$ is a probability density function on the interval $[1, 2]$. If this is not possible, explain why.

Solution: To be a probability density function, $f(x)$ must be non-negative on the interval $[1, 2]$ and integrate to 1 over this interval. The constant k will be determined by this second condition, after which we can check that the first condition is also satisfied.

We integrate $f(x)$ from $x = 1$ to $x = 2$ in order to determine k:

$$1 = \int_1^2 f(x)\,dx = \int_1^2 \frac{k}{x^4}\,dx = k\int_1^2 x^{-4}\,dx = k\left(\frac{1}{-3}x^{-3}\right)\Big|_1^2.$$

We evaluate this at the endpoints to get

$$1 = k\left(-\frac{1}{3x^3}\right)\Big|_1^2 = k\left[-\frac{1}{3\cdot 2^3} - \left(-\frac{1}{3\cdot 1^3}\right)\right] = k\left(-\frac{1}{24}+\frac{1}{3}\right) = \frac{7}{24}\cdot k.$$

Thus $k = \frac{24}{7} \approx 3.4286$.

Notice that k is positive, so $f(x)$ is positive on the interval $[1, 2]$ as well. Thus the first condition we listed is automatically satisfied by $f(x)$ for this choice of k.

5 A number x is selected at random from the interval $[0, 4]$. The probability density function for x is

$$f(x) = \frac{1}{8}x \qquad \text{for } \ 0 \le x \le 4.$$

Find the probability that a number is selected in the subinterval $[3, 4]$.

Solution: If we let X be the random variable that is described by this problem, then the question asks for $\Pr(3 \le X \le 4)$. Since X is a continuous random variable with probability density function $f(x)$, this probability is

$$\Pr(3 \le X \le 4) = \int_3^4 f(x)\,dx = \int_3^4 \frac{1}{8}x\,dx.$$

This is just a computation:

$$\Pr(3 \le X \le 4) = \int_3^4 \frac{1}{8}x\,dx = \frac{1}{8}\cdot\frac{1}{2}x^2\Big|_3^4 = \frac{1}{16}\left(4^2 - 3^2\right) = \frac{7}{16}.$$

Thus the probability that the selected number is in the given subinterval is $\frac{7}{16}$ or 43.75%.