Quiz Four Solutions

1 Suppose a car rental agency rents cars for \$30 per day. The cars cost the agency \$15 per day. Suppose that the agency has records that indicate a typical day has 9 to 11 customers, with probabilities according to the following table:

Number of Customers	9	10	11
Probability	0.3	0.3	0.4

How many cars should the rental agency have on hand in order to maximize their profits?

Solution: We'll compute the expected profit for each of the reasonable possibilities available: 9 cars, 10 cars, or 11 cars. (It's unreasonable to have fewer than 9 cars, since we can always rent out at least 9. Similarly, since we never have more than 11 customers, it's unreasonable to have more than 11 cars.)

Before we compute the expected profit in each case, let's mention the profit per car per day. We either lose \$15 per car per day (if a particular car is not rented) or make a profit of $\$30 - \$15 = \$15$ per car per day if it is rented. That is, the daily profit or loss for a single car is

$$\begin{cases} \$15 \text{ profit} & \text{if car is rented} \\ \$15 \text{ loss} & \text{if car is } \textit{not} \text{ rented.} \end{cases}$$

So what's the expected profit if the agency has 9 cars? In this case, all the cars will be rented every day, so the expected profit is (and I've put a 9 subscript here to distinguish it from later expected values):

$$E_9 = 9 \cdot \$15 = \$135 \text{ per day.}$$

What's the expected profit if the agency has 10 cars? Now we rent 9 cars with probability 0.3 and 10 cars with probability 0.7 (when we have 10 or 11 customers). Thus the expected profit (if the agency has 10 cars – hence the 10 subscript) is

$$E_{10} = 0.3\big(9 \cdot \$15 + 1 \cdot (-\$15)\big) + 0.7\,(10 \cdot \$15) = \$141.$$

Finally, what is the expected profit if the agency has 11 cars? This is the most complicated: with probability 0.3 we can rent 9 cars, with probability 0.3 we can rent 10, and with probability 0.4 we can rent 11. Thus the expected profit (if the agency has 11 cars) is

$$E_{11} = 0.3\big(9 \cdot \$15 + 2 \cdot (-\$15)\big) + 0.3\big(10 \cdot \$15 + 1 \cdot (-\$15)\big) + 0.4\,(11 \cdot \$15) = \$138$$

Thus we have a small table:

Number of Cars	9	10	11
Expected Profit	\$135	\$141	\$138

Thus, to maximize the expected profit, we should stock our agency with 10 cars.

2 A large urn contains 25 balls: 20 are colored blue and 5 are colored yellow. A ball is drawn, its color is noted, then the ball is replaced. This is done 3 times.

(a) Let X be the random variable representing the number of yellow balls picked in the three drawings. List the values of this random variable together with the probability distribution.

Solution: The possible values of the random variable X are 0, 1, 2, or 3 (the number of yellow balls). The probability that X is, say, 1, is the probability of drawing 1 yellow ball in three attempts. Since each draw has the same probability of obtaining a yellow ball (namely $p = \frac{5}{25} = 0.2$), this is a Bernoulli trial and thus has probability

$$b(3, 1; 0.2) = \binom{3}{1}(0.2)^1(0.8)^2 = 0.384.$$

We could compute the other probabilities similarly; instead we simply fill out a small table:

k	$\Pr(X = k)$
0	$b(3, 0; 0.2) = 0.512$
1	$b(3, 1; 0.2) = 0.384$
2	$b(3, 2; 0.2) = 0.096$
3	$b(3, 3; 0.2) = 0.008$

(b) Find $E(X)$, the expected value of the random variable you found in part (a).

Solution: While we could find $E(X)$ using the probabilities in the table, above, it is simpler to use a rule for Bernoulli trials. If X is a random variable representing the results of Bernoulli trial with n repetitions and probability of success p, then $E(X) = n \cdot p$. Here $n = 3$ and $p = 0.2$ (the probability of a yellow ball), so $E(X) = n \cdot p = 3 \cdot 0.2 = 0.6$.

3 A wallet contains seven bills: two \$1 bills, two \$5 bills, a \$10 bill, and two \$20 bills. Suppose we take out one of the bills randomly. What is the expected value of this bill?

Solution: Each bill is selected with equal probability, namely one-seventh. Thus the expected value is

$$E = \frac{1}{7}\Big(2 \cdot \$1 + 2 \cdot \$5 + \$10 + 2 \cdot \$20\Big) = \frac{\$62}{7} = \$8\,{}^{6}\!/_{7} \approx \$8.86.$$