1. Find the value, if any, of each of the following improper integrals. You must use limits to justify your answers!

(a) $\int_0^\infty e^{-5x}\,dx$

Solution: The improper integral is a limit:

$$\int_0^\infty e^{-5x}\,dx = \lim_{b\to\infty}\int_0^b e^{-5x}\,dx.$$

We can then compute the definite integral explicitly:

$$\int_0^\infty e^{-5x}\,dx = \lim_{b\to\infty}\int_0^b e^{-5x}\,dx = \lim_{b\to\infty}\left(-\frac{1}{5}e^{-5}\Big|_0^b\right)$$

$$= -\frac{1}{5}\lim_{b\to\infty}\left(e^{-5b} - e^0\right) = -\frac{1}{5}\lim_{b\to\infty}\left(e^{-5b} - 1\right).$$

As b tends to ∞, $-5b$ tends to $-\infty$ and so e^{-5b} tends to 0. Thus

$$\int_0^\infty e^{-5x}\,dx = -\frac{1}{5}\lim_{b\to\infty}\left(e^{-5b} - 1\right) = -\frac{1}{5}(0-1) = \frac{1}{5}.$$

This is our final answer.

(b) $\int_0^1 \frac{1}{\sqrt{x}}\,dx$

Solution: This integral is improper because the integrand is undefined at $x = 0$. To compute, we again write the integral as a limit:

$$\int_0^1 \frac{1}{\sqrt{x}}\,dx = \lim_{a\to 0^+}\int_a^1 x^{-1/2}\,dx = \lim_{a\to 0^+}\left(2x^{1/2}\Big|_a^1\right) = \lim_{a\to 0^+}\left(2\sqrt{1} - 2\sqrt{a}\right).$$

Now as a decreases to zero, $2\sqrt{a}$ shrinks to zero, and so

$$\int_0^1 \frac{1}{\sqrt{x}}\,dx = \lim_{a\to 0^+}\left(2\sqrt{1} - 2\sqrt{a}\right) = 2 - 0 = 2.$$

Thus the integral has value 2.

(c) $\displaystyle\int_1^\infty \frac{1}{\sqrt{x}}\,dx$

Solution: This is the same integrand as in part (b), but with different limits of integration. This integral is divergent. To see why, we again write the integral as a limit and compute:

$$\int_1^\infty \frac{1}{\sqrt{x}}\,dx = \lim_{b\to\infty}\int_1^b x^{-1/2}\,dx = \lim_{b\to\infty}\left(2\sqrt{x}\Big|_1^b\right) = \lim_{b\to\infty}\left(2\sqrt{b} - 2\sqrt{1}\right).$$

Now as b grows without bound, $2\sqrt{b}$ does as well. Thus the limit as b goes to infinity of $2\sqrt{b}$ is infinite, and so

$$\int_1^\infty \frac{1}{\sqrt{x}}\,dx = \lim_{b\to\infty}\left(2\sqrt{b} - 2\sqrt{1}\right) = \infty - 2 = \infty.$$

Thus the integral is divergent.

2 Find the area, if it exists, under the curve $y = \dfrac{1}{x^2}$ to the right of $x = 2$.

Solution: This area is represented by the improper integral

$$\text{Area} = \int_2^\infty \frac{1}{x^2}\,dx.$$

So the question becomes: does this improper integral have a (finite) value? We compute this, as usual, with limits:

$$\int_2^\infty \frac{1}{x^2}\,dx = \lim_{b\to\infty}\int_2^b x^{-2}\,dx = \lim_{b\to\infty}\left(-x^{-1}\Big|_2^b\right) = \lim_{b\to\infty}\left[-\frac{1}{b} - \left(-\frac{1}{2}\right)\right].$$

As b increases without bound, $\frac{1}{b}$ shrinks to zero. This means that the first term in the limit is zero, and so

$$\int_2^\infty \frac{1}{x^2}\,dx = \lim_{b\to\infty}\left(-\frac{1}{b} + \frac{1}{2}\right) = 0 + \frac{1}{2} = \frac{1}{2}.$$

Put another way, the area under the curve to the right of $x = 2$ is $\frac{1}{2}$.

3 Suppose that 5% of the items produced by a factory are defective. If 12 items are chosen at random, what is the probability that...

(For this problem you need not simplify. That is, your final answer may involve expressions such as $\binom{5}{3}$ and $(0.4)^3$. Your final answer should **not** include the expression $b(n,k;p)$.)

(a) ...exactly 1 is defective?

Solution: This is a Bernoulli trial where a "success" is a defective item; the probability that a single item is defective is $p = 0.05$. We're asked for the probability of exactly $k = 1$ success in $n = 12$ trials, which is

$$b(12, 1, 0.05) = \binom{12}{1}(0.05)^1(0.95)^{11} \approx 0.34128005536588.$$

Either of the two right-most answers is enough.

(b) ...at least 1 is defective?

Solution: This is the probability at least 1 "successes" (using the same notation as part (a)). This is most easily found using the complement: no "successes" is the complement of at least 1 success. Thus the probability of at least 1 success is

$$1 - b(12, 0; 0.05) = 1 - \binom{12}{0}(0.05)^0(0.95)^{12} = 1 - (0.95)^{12} \approx 0.45963991233736.$$

Any answer after the first is enough.

(c) ...2 or fewer are defective?

Solution: Two or fewer "successes" is 0, 1, or 2 successes. Thus the probability asked for is

$$\begin{aligned}
&b(12, 0; 0.05) + b(12, 1; 0.05) + b(12, 2; 0.05)\\
&= \binom{12}{0}(0.05)^0(0.95)^{12} + \binom{12}{1}(0.05)^1(0.95)^{11} + \binom{12}{2}(0.05)^2(0.95)^{10}\\
&= (1)(1)(0.95)^{12} + 12(0.05)(0.95)^{11} + 66(0.05)^2(0.95)^{10}\\
&\approx 0.98043173800285.
\end{aligned}$$

Again, any answer after the first is enough.