

1 Find the value, if any, of each of the following improper integrals. You must use limits to justify your answers!

(a)  $\int_0^{\infty} e^{-3x} dx$

**Solution:** The improper integral is a limit:

$$\int_0^{\infty} e^{-3x} dx = \lim_{b \rightarrow \infty} \int_0^b e^{-3x} dx.$$

We can then compute the definite integral explicitly:

$$\begin{aligned} \int_0^{\infty} e^{-3x} dx &= \lim_{b \rightarrow \infty} \int_0^b e^{-3x} dx = \lim_{b \rightarrow \infty} \left( -\frac{1}{3} e^{-3x} \Big|_0^b \right) \\ &= -\frac{1}{3} \lim_{b \rightarrow \infty} (e^{-3b} - e^0) = -\frac{1}{3} \lim_{b \rightarrow \infty} (e^{-3b} - 1). \end{aligned}$$

As  $b$  tends to  $\infty$ ,  $-3b$  tends to  $-\infty$  and so  $e^{-3b}$  tends to 0. Thus

$$\int_0^{\infty} e^{-3x} dx = -\frac{1}{3} \lim_{b \rightarrow \infty} (e^{-3b} - 1) = -\frac{1}{3} (0 - 1) = \frac{1}{3}.$$

This is our final answer.

(b)  $\int_0^1 \frac{1}{x^{3/2}} dx$

**Solution:** This improper integral is divergent. (The integral is improper because the integrand is undefined at  $x = 0$ .) To see why, let's try to compute the integral:

$$\begin{aligned} \int_0^1 \frac{1}{x^{3/2}} dx &= \lim_{a \rightarrow 0^+} \int_a^1 x^{-3/2} dx = \lim_{a \rightarrow 0^+} \left( -\frac{1}{1/2} x^{-1/2} \Big|_a^1 \right) = \lim_{a \rightarrow 0^+} \left( -\frac{2}{\sqrt{x}} \Big|_a^1 \right) \\ &= \lim_{a \rightarrow 0^+} \left[ -\frac{2}{\sqrt{1}} - \left( -\frac{2}{\sqrt{a}} \right) \right] = \lim_{a \rightarrow 0^+} \left( -2 + \frac{2}{\sqrt{a}} \right). \end{aligned}$$

As  $a$  decreases to zero,  $\frac{2}{\sqrt{a}}$  grows to  $+\infty$ . Thus this limit (and hence the integral) is not finite. That is, the integral diverges.

$$(c) \int_1^{\infty} \frac{1}{x^{3/2}} dx$$

**Solution:** This is the same integrand as in part (b), but with different limits of integration. We again write the integral as a limit and compute:

$$\int_1^{\infty} \frac{1}{x^{3/2}} dx = \lim_{b \rightarrow \infty} \int_1^b x^{-3/2} dx = \lim_{b \rightarrow \infty} \left( -\frac{2}{\sqrt{x}} \Big|_1^b \right) = \lim_{b \rightarrow \infty} \left[ -\frac{2}{\sqrt{b}} - \left( -\frac{2}{\sqrt{1}} \right) \right].$$

Now as  $b$  grows without bound,  $\frac{2}{\sqrt{b}}$  shrinks to zero. Thus the limit as  $b$  goes to infinity of  $\frac{2}{\sqrt{b}}$  is zero, and so

$$\int_1^{\infty} \frac{1}{x^{3/2}} dx = \lim_{b \rightarrow \infty} \left[ -\frac{2}{\sqrt{b}} - \left( -\frac{2}{\sqrt{1}} \right) \right] = 0 + 2 = 2.$$

Thus the integral has value 2.

- 2 Find the area, if it exists, under the curve  $y = \frac{1}{x^2}$  between  $x = 0$  and  $x = 2$ .

**Solution:** This area is represented by the improper integral

$$\text{Area} = \int_0^2 \frac{1}{x^2} dx.$$

So the question becomes: does this improper integral have a (finite) value? We compute this, as usual, with limits:

$$\int_0^2 \frac{1}{x^2} dx = \lim_{a \rightarrow 0^+} \int_a^2 x^{-2} dx = \lim_{a \rightarrow 0^+} \left( -x^{-1} \Big|_a^2 \right) = \lim_{a \rightarrow 0^+} \left[ -\frac{1}{2} - \left( -\frac{1}{a} \right) \right].$$

As  $a$  decreases to zero,  $\frac{1}{a}$  grows without bound. This means that the area is infinite:

$$\int_0^2 \frac{1}{x^2} dx = \lim_{a \rightarrow 0^+} \left( -\frac{1}{2} + \frac{1}{a} \right) = -\frac{1}{2} + \infty = \infty.$$

Put another way, the area is not defined as a finite number.

- 3 Suppose that 4% of the items produced by a factory are defective. If 10 items are chosen at random, what is the probability that...

(For this problem you need not simplify. That is, your final answer may involve expressions such as  $\binom{5}{3}$  and  $(0.4)^3$ . Your final answer should **not** include the expression  $b(n, k; p)$ .)

- (a) ... exactly 1 is defective?

**Solution:** This is a Bernoulli trial where a “success” is a defective item; the probability that a single item is defective is  $p = 0.04$ . We’re asked for the probability of exactly  $k = 1$  success in  $n = 10$  trials, which is

$$b(10, 1, 0.04) = \binom{10}{1}(0.04)^1(0.96)^9 \approx 0.27701359832979.$$

Either of the two right-most answers is enough.

- (b) ... at least 1 is defective?

**Solution:** This is the probability at least 1 “successes” (using the same notation as part (a)). This is most easily found using the complement: no “successes” is the complement of at least 1 success. Thus the probability of at least 1 success is

$$1 - b(10, 0; 0.04) = 1 - \binom{10}{0}(0.04)^0(0.96)^{10} = 1 - (0.96)^{10} \approx 0.3351673640085.$$

Any answer after the first is enough.

- (c) ... 2 or fewer are defective?

**Solution:** Two or fewer “successes” is 0, 1, or 2 successes. Thus the probability asked for is

$$\begin{aligned} & b(10, 0; 0.04) + b(10, 1; 0.04) + b(10, 2; 0.04) \\ &= \binom{10}{0}(0.04)^0(0.96)^{10} + \binom{10}{1}(0.04)^1(0.96)^9 + \binom{10}{2}(0.04)^2(0.96)^8 \\ &= (1)(1)(0.96)^{10} + 10(0.04)(0.96)^9 + 45(0.04)^2(0.96)^8 \\ &\approx 0.99378628400813. \end{aligned}$$

Again, any answer after the first is enough.