Quiz Two Solutions

1 Compute the following integrals.

(a) $\int t^2 \sin(t)\, dt$

Solution: This integral requires that we integrate by parts twice. The first time we choose $u = t^2$ and $dv = \sin t\, dt$, so $du = 2t\, dt$ and $v = -\cos(t)$. Thus

$$\int t^2 \sin(t)\, dt = uv - \int v\, du = t^2\left(-\cos(t)\right) - \int (-\cos(t)) \cdot 2t\, dt = -t^2\cos(t) + 2\int t\cos(t)\, dt.$$

Notice we've made the integral simpler: it's changed from $\int t^2 \sin(t)\, dt$ to $\int t\cos(t)\, dt$. We compute this second integral using a similar choice of u and dv: $u = t$ and $dv = \cos(t)\, dt$, so $du = dt$ and $v = \sin(t)$. Thus

$$\begin{aligned}\int t\cos(t)\, dt &= uv - \int v\, du = t\sin(t) - \int \sin(t)\, dt \\ &= t\sin(t) - (-\cos(t)) + K = t\sin(t) + \cos(t) + K.\end{aligned}$$

Combining these two computations gives us

$$\int t^2 \sin(t)\, dt = -t^2\cos(t) + 2\left(t\sin(t) + \cos(t) + K\right) = -t^2\cos(t) + 2t\sin(t) + 2\cos(t) + K_1,$$

where again K_1 is an unknown constant (that is slightly different than K).

(b) $\int_0^2 \frac{x}{x^2+1}\, dx$

Solution: In order to compute this definite integral, we need a substitution. We make the substitution $u = x^2+1$ (aiming towards the integral $\int \frac{1}{u}\, du = \ln(|u|) + K$), so $\frac{du}{dx} = 2x$ or $du = 2x\, dx$ and $dx = \frac{1}{2x}\, du$. Thus

$$\int_0^2 \frac{x}{x^2+1}\, dx = \int_{u=1}^{u=5} \frac{x}{u} \cdot \frac{1}{2x}\, du = \frac{1}{2}\int_1^5 \frac{1}{u}\, du.$$

Notice that we've changed the limits of integration. Recall that $u = x^2+1$, so we compute the new limits as follows: If $x = 0$, then $u = 0^2 + 1 = 1$. If $x = 2$, then $u = 2^2 + 1 = 5$. Thus our integral is

$$\int_0^2 \frac{x}{x^2+1}\, dx = \frac{1}{2}\int_1^5 \frac{1}{u}\, du = \frac{1}{2}\ln\left(|u|\right)\Big|_1^5 = \frac{1}{2}\ln(5) - \frac{1}{2}\ln(1) = \frac{1}{2}\ln(5)$$

(since $\ln(1) = 0$). This is our final answer: $\displaystyle\int_0^2 \frac{x}{x^2+1}\, dx = \frac{1}{2}\ln(5) \approx 0.8047.$

2 The depreciation rate of a house in today's housing market obeys the equation $V'(t) = -12,000te^{-2t}$, where $V(t)$ is the value of the house in t years. If the house is currently worth \$180,000, what will it be worth in two years? Please round your answer to the nearest dollar.

Solution: We calculate $V(t)$ by integrating $V'(t) = -12,000te^{-2t}$ and using the initial condition $V(0) = \$180,000$. The integral requires that we integrate by parts:

$$V(t) = \int -12,000te^{-2t}\,dt = -12,000 \int te^{-2t}\,dt.$$

We let $u = t$ and $dv = e^{-2t}\,dt$, so $du = dt$ and $v = -\frac{1}{2}e^{-2t}$. Plugging these in, we get

$$\begin{aligned}\int te^{-2t}\,dt &= t \cdot -\frac{1}{2}e^{-2t} - \int -\frac{1}{2}e^{-2t}\,dt \\ &= -\frac{1}{2}te^{-2t} + \frac{1}{2}\int e^{-2t}\,dt \\ &= -\frac{1}{2}te^{-2t} - \frac{1}{4}e^{-2t} + K.\end{aligned}$$

Thus

$$\begin{aligned}V(t) &= -12,000\left(-\frac{1}{2}te^{-2t} - \frac{1}{4}e^{-2t} + K\right) \\ &= 6,000te^{-2t} + 3,000e^{-2t} + C,\end{aligned}$$

where $C = -12,000K$ is an unknown constant. We find C using $V(0) = \$180,000$:

$$180,000V(0) = 6,000(0)e^{-2(0)} + 3,000e^{-2(0)} + C = 3,000 + C.$$

Thus $C = 177,000$, so the value of the house in two years is

$$V(2) = 6,000(2)e^{-2(2)} + 3,000e^{-2(2)} + 177,000 = 15,000e^{-4} + 177,000 \approx \$177,275.$$

This is our final answer.

3 The marginal cost function for producing x units is $9x^2 - 50x + 200$ dollars. Find the increase in cost if production is increased from 100 to 105 units.

Solution: The increase in the cost if production is increased from 100 to 105 units is the definite integral of the marginal cost from 100 to 105. That is, we're asked to find

$$\int_{100}^{105} \left(9x^2 - 50x + 200\right)\, dx.$$

This is straightforward, and we do the computation without comment:

$$\begin{aligned}\int_{100}^{105} \left(9x^2 - 50x + 200\right)\, dx &= \left(3x^3 - 25x^2 + 200x\right)\Big|_{100}^{105} \\ &= \left(3(105)^3 - 25(105)^2 + 200(105)\right) - \left(3(100)^3 - 25(100)^2 + 200(100)\right) \\ &= 3,218,250 - 2,770,000 = 448,250.\end{aligned}$$

Thus our final answer is that the increase in cost is \$448,250.

Another approach to evaluating this definite integral is to calculate term-by-term. That is, we could have written the following for the above integral:

$$\begin{aligned}\int_{100}^{105} \left(9x^2 - 50x + 200\right)\, dx &= \left(3x^3 - 25x^2 + 200x\right)\Big|_{100}^{105} \\ &= 3\left((105)^3 - (100)^3\right) - 25\left((105)^2 - (100)^2\right) + 200\,(105 - 100) \\ &= 3 \cdot 157,625 - 25 \cdot 1,025 + 200 \cdot 5 \\ &= 472,875 - 25,625 + 1,000 = 448,250.\end{aligned}$$

This is, of course, our previous answer.