

1 Find the following indefinite integrals. Show your method for each part.

$$(a) \int (2x^{-2} - e^{3x}) dx = -2x^{-1} - \frac{1}{3}e^{3x} + K$$

**Solution:** This uses the two rules for integration:

$$\int x^n dx = \frac{1}{n+1}x^{n+1} + K \quad (\text{if } n \neq -1) \quad \text{and} \quad \int e^{kx} dx = \frac{1}{k}e^{kx} + K.$$

From this we get

$$\begin{aligned} \int (2x^{-2} - e^{3x}) dx &= 2 \int x^{-2} dx - \int e^{3x} dx \\ &= 2 \left( \frac{1}{-1}x^{-1} + K_1 \right) - \left( \frac{1}{3}e^{3x} + K_2 \right) \\ &= -2x^{-1} - \frac{1}{3}e^{3x} + K. \end{aligned}$$

$$(b) \int \frac{(\ln x)^5}{2x} dx = \frac{1}{12}(\ln x)^6 + K$$

**Solution:** Let  $u = \ln(x)$ , so  $u' = \frac{1}{x}$  and  $du = \frac{1}{x} dx$ . Thus we have

$$\begin{aligned} \int \frac{(\ln x)^5}{2x} dx &= \frac{1}{2} \int (\ln x)^5 \cdot \frac{1}{x} dx \\ &= \frac{1}{2} \int u^5 du \\ &= \frac{1}{2} \cdot \frac{1}{6} u^6 + K \\ &= \frac{1}{12} (\ln x)^6 + K \end{aligned}$$

$$(c) \int 2\sqrt{x} \ln(x) dx = \frac{4}{3}x^{3/2} \ln(x) - \frac{8}{9}x^{3/2} + K$$

**Solution:** Here we use integration by parts: let  $u = \ln(x)$  and  $dv = \sqrt{x} dx$ . Note that we can integrate  $dv$  to get an anti-derivative  $v = \frac{2}{3}x^{3/2}$ , and that  $du = \frac{1}{x} dx$  (so that we no longer have a troublesome logarithm to integrate). Using the integration by parts expression

$$\int u dv = uv - \int v du,$$

we get

$$\begin{aligned} 2 \int \sqrt{x} \ln(x) dx &= 2 \ln(x) \frac{2}{3}x^{3/2} - 2 \int \frac{2}{3}x^{3/2} \frac{1}{x} dx \\ &= \frac{4}{3}x^{3/2} \ln(x) - \frac{4}{3} \int x^{1/2} dx \\ &= \frac{4}{3}x^{3/2} \ln(x) - \frac{4}{3} \cdot \frac{2}{3}x^{3/2} + K \\ &= \frac{4}{3}x^{3/2} \ln(x) - \frac{8}{9}x^{3/2} + K. \end{aligned}$$

$$(d) \int \frac{x}{\sqrt{1-x}} dx = -\frac{4}{3}\sqrt{1-x} + \frac{2}{3}x\sqrt{1-x} + K$$

**Solution:** There are several ways to do this. One simple way involves the substitution  $u = 1 - x$ , or  $x = 1 - u$ , so  $dx = -du$ . Thus the integral we're looking for is

$$\begin{aligned} \int \frac{x}{\sqrt{1-x}} dx &= -\int \frac{1-u}{\sqrt{u}} du \\ &= -\int u^{-1/2} du + \int u^{1/2} du \\ &= -2u^{1/2} + \frac{2}{3}u^{3/2} + K \\ &= -2\sqrt{1-x} + \frac{2}{3}(1-x)^{3/2} + K. \end{aligned}$$

This can also be written as

$$\begin{aligned} -2\sqrt{1-x} + \frac{2}{3}(1-x)\sqrt{1-x} + K &= \left(-2 + \frac{2}{3}\right)\sqrt{1-x} + \frac{2}{3}x\sqrt{1-x} + K \\ &= -\frac{4}{3}\sqrt{1-x} + \frac{2}{3}x\sqrt{1-x} + K, \end{aligned}$$

which may look cleaner.

$$(e) \int \left(3x^4 - \frac{\sqrt{x}}{3} + e^{-3x}\right) dx = \frac{3}{5}x^5 - \frac{2}{9}x^{3/2} - \frac{1}{3}e^{-3x} + K$$

**Solution:** We compute:

$$\begin{aligned} \int \left(3x^4 - \frac{\sqrt{x}}{3} + e^{-3x}\right) dx &= 3 \int x^4 dx - \frac{1}{3} \int x^{1/2} dx + \int e^{-3x} dx \\ &= 3 \frac{1}{5}x^5 - \frac{1}{3} \cdot \frac{1}{3/2}x^{3/2} + \frac{1}{-3}e^{-3x} + K \\ &= \frac{3}{5}x^5 - \frac{2}{9}x^{3/2} - \frac{1}{3}e^{-3x} + K. \end{aligned}$$

$$(f) \int 3xe^{2x} dx = \frac{3}{2}xe^{2x} - \frac{3}{4}e^{2x} + K$$

**Solution:** For this we integrate by parts. Let  $u = 3x$ , and  $dv = e^{2x} dx$ . Then we have

$$\begin{aligned} u &= 3x & dv &= e^{2x} dx \\ du &= 3 dx & v &= \frac{1}{2}e^{2x}. \end{aligned}$$

We then have

$$\begin{aligned} \int 3xe^{2x} dx &= 3x \cdot \frac{1}{2}e^{2x} - \int \frac{1}{2}e^{2x} \cdot 3 dx \\ &= \frac{3}{2}xe^{2x} - \frac{3}{2} \int e^{2x} dx \\ &= \frac{3}{2}xe^{2x} - \frac{3}{2} \cdot \frac{1}{2}e^{2x} + K \\ &= \frac{3}{2}xe^{2x} - \frac{3}{4}e^{2x} + K. \end{aligned}$$

$$(g) \int (e^{-2x} - x^{-1} + x^{-2}) dx = -\frac{1}{2}e^{-2x} - \ln(|x|) - x^{-1} + K$$

**Solution:** This is a computation using the following basic rules of integration:

$$\int e^{kx} dx = \frac{1}{k}e^{kx} + K \quad \int x^k dx = \begin{cases} \frac{1}{k+1}x^{k+1} + K & \text{if } k \neq -1 \\ \ln(|x|) + K & \text{if } k = -1 \end{cases}$$

Thus we have

$$\begin{aligned} \int (e^{-2x} - x^{-1} + x^{-2}) dx &= \int e^{-2x} dx - \int x^{-1} dx + \int x^{-2} dx \\ &= \frac{1}{-2}e^{-2x} - \ln(|x|) + \frac{1}{-2+1}x^{-2+1} + K \\ &= -\frac{1}{2}e^{-2x} - \ln(|x|) - x^{-1} + K. \end{aligned}$$

$$(h) \int \frac{x}{1+3x^2} dx = \frac{1}{6} \ln(1+3x^2) + K$$

**Solution:** Here we use substitution, making the change  $u = 1 + 3x^2$ . Then  $du = u' dx = 6x dx$ , so  $x dx = \frac{1}{6}du$ . That is,

$$\begin{aligned} \int \frac{x}{1+3x^2} dx &= \int (1+3x^2)^{-1} x dx \\ &= \int u^{-1} \frac{1}{6} du = \frac{1}{6} \int u^{-1} du \\ &= \frac{1}{6} \ln(|u|) + K \\ &= \frac{1}{6} \ln(1+3x^2) + K. \end{aligned}$$

(Notice that  $|1+3x^2| = 1+3x^2$ , since  $x^2 \geq 0$ .)

$$(i) \int xe^{4x} dx = \frac{1}{4}xe^{4x} - \frac{1}{16}e^{4x} + K$$

**Solution:** This problem requires us to use integration by parts. We let

$$\begin{aligned} u &= x & dv &= e^{4x} dx \\ du &= dx & v &= \frac{1}{4}e^{4x}. \end{aligned}$$

Using this substitution, we get

$$\begin{aligned} \int xe^{4x} dx &= x \cdot \frac{1}{4}e^{4x} - \int \frac{1}{4}e^{4x} dx \\ &= \frac{1}{4}xe^{4x} - \frac{1}{4} \cdot \frac{1}{4}e^{4x} + K \\ &= \frac{1}{4}xe^{4x} - \frac{1}{16}e^{4x} + K. \end{aligned}$$

One can rewrite this to  $\frac{1}{4}e^{4x}(x - \frac{1}{4}) + K$ , but this is not necessary.

2 Compute the following definite integrals. Show your solution method.

$$(a) \int_1^2 \left( -\frac{4}{3x} - 2x^{-3} \right) dx = -\frac{4}{3} \ln(2) - \frac{3}{4}$$

**Solution:** Using our rules for integration, we compute:

$$\begin{aligned} \int_1^2 \left( -\frac{4}{3x} - 2x^{-3} \right) dx &= -\frac{4}{3} \int_1^2 x^{-1} dx - 2 \int_1^2 x^{-3} dx \\ &= -\frac{4}{3} \ln(x) \Big|_1^2 - 2 \left[ \frac{1}{-2} x^{-2} \right]_1^2 \\ &= -\frac{4}{3} [\ln(2) - \ln(1)] + \left[ \frac{1}{2^2} - \frac{1}{1^2} \right] \\ &= -\frac{4}{3} \ln(2) - \frac{3}{4}. \end{aligned}$$

$$(b) \int_0^{\pi/2} \sin^2(2x) \cos(2x) dx = 0$$

**Solution:** Here we make the substitution  $u = \sin(2x)$ . From this we get  $du = 2 \cos(2x) dx$ , or  $\cos(2x) dx = \frac{1}{2} du$ . Substituting this into the integral gives us

$$\begin{aligned} \int_0^{\pi/2} \sin^2(2x) \cos(2x) dx &= \int_{x=0}^{x=\pi/2} u^2 \cdot \frac{1}{2} du \\ &= \frac{1}{6} u^3 \Big|_{x=0}^{x=\pi/2} \\ &= \frac{1}{6} \sin^3(2x) \Big|_{x=0}^{x=\pi/2} \\ &= \frac{1}{6} \sin^3(\pi) - \frac{1}{6} \sin^3(0) = 0. \end{aligned}$$

If we had made the substitution in the limits as well, we would have seen that the answer is zero earlier. When  $x = 0$ ,  $u = \sin(0) = 0$ ; and when  $x = \pi/2$ ,  $u = \sin(\pi) = 0$  as well. Thus, after making the substitution, the integral is

$$\int_{u=0}^{u=0} u^2 \frac{1}{2} du.$$

Since the endpoints of the integral are the same, the definite integral must be zero.

$$(c) \int_0^{2\pi} x \cos(x^2 - \pi) dx = \frac{1}{2} \sin(4\pi^2 - \pi)$$

**Solution:** Here make the substitution  $u = x^2 - \pi$ . This means that  $du = 2x dx$ , so  $x dx = \frac{1}{2} du$ . When  $x = 0$ ,  $u = -\pi$ ; when  $x = 2\pi$ ,  $u = 4\pi^2 - \pi$ . Thus the integral becomes

$$\begin{aligned} \int_0^{2\pi} x \cos(x^2 - \pi) dx &= \int_{-\pi}^{4\pi^2 - \pi} \cos(u) \frac{1}{2} du \\ &= \frac{1}{2} \sin(u) \Big|_{-\pi}^{4\pi^2 - \pi} \\ &= \frac{1}{2} (\sin(4\pi^2 - \pi) - \sin(-\pi)) \\ &= \frac{1}{2} \sin(4\pi^2 - \pi) \end{aligned}$$

$$(d) \int_1^2 \frac{9x^2 + 12x}{x^3 + 2x^2 - 2} dx = 3 \ln(14)$$

**Solution:** We make the substitution  $u = x^3 + 2x^2 - 2$ . Then  $u' = 3x^2 + 4x$ , and  $du = (3x^2 + 4x) dx$ . For the limits of integration, notice that  $u = 1$  when  $x = 1$  and  $u = 14$  when  $x = 2$ . We can then integrate:

$$\begin{aligned} \int_1^2 \frac{9x^2 + 12x}{x^3 + 2x^2 - 2} dx &= \int_1^2 \frac{3(3x^2 + 4x)}{x^3 + 2x^2 - 2} dx \\ &= 3 \int_1^{14} \frac{du}{u} \\ &= 3 \ln(u) \Big|_1^{14} = 3 \ln(14) - 3 \ln(1) = 3 \ln(14). \end{aligned}$$

$$(e) \int_0^{\pi/2} x \sin(8x) dx = -\frac{\pi}{16}$$

**Solution:** To find the indefinite integral (or anti-derivative) we use integration by parts:

$$\begin{aligned} u &= x & dv &= \sin(8x) dx \\ du &= dx & v &= -\frac{1}{8} \cos(8x). \end{aligned}$$

Integrating, we see that:

$$\begin{aligned} \int x \sin(8x) dx &= x \cdot -\frac{1}{8} \cos(8x) - \int -\frac{1}{8} \cos(8x) dx \\ &= -\frac{1}{8} x \cos(8x) + \frac{1}{8} \int \cos(8x) dx \\ &= -\frac{1}{8} x \cos(8x) + \frac{1}{8} \cdot \frac{1}{8} \sin(8x) + K \\ &= \frac{1}{8} x \cos(8x) + \frac{1}{64} \sin(8x) + K. \end{aligned}$$

Since we're computing a *definite* integral, we can use any choice of anti-derivative we wish, so we choose the one with  $K = 0$ . Thus we have

$$\begin{aligned} \int_0^{\pi/2} x \sin(8x) dx &= \left( -\frac{1}{8} x \cos(8x) + \frac{1}{64} \sin(8x) \right) \Big|_0^{\pi/2} \\ &= \left( -\frac{1}{8} \frac{\pi}{2} \cos(4\pi) + \frac{1}{64} \sin(4\pi) \right) - \left( -\frac{1}{8} 0 \cos(0) + \frac{1}{64} \sin(0) \right) \\ &= \left( -\frac{\pi}{16} \cdot 1 + \frac{1}{64} \cdot 1 \right) - \left( 0 + \frac{1}{64} \cdot 1 \right) \\ &= -\frac{\pi}{16}. \end{aligned}$$

Thus the definite integral is  $-\frac{\pi}{16}$ .

$$(f) \int_1^e \left( \frac{2}{x} - x^2 \right) dx = \frac{7}{3} - \frac{1}{3} e^3 \approx -4.36$$

**Solution:** Here the integration is straightforward:

$$\begin{aligned} \int_1^e \left( \frac{2}{x} - x^2 \right) dx &= \left[ 2 \ln(|x|) - \frac{1}{3} x^3 \right]_1^e \\ &= \left( 2 \ln(e) - \frac{1}{3} e^3 \right) - \left( 2 \ln(1) - \frac{1}{3} 1^3 \right) \\ &= \frac{7}{3} - \frac{1}{3} e^3 \approx -4.36 \end{aligned}$$

$$(g) \int_0^{\pi/2} x \cos(x) dx = \frac{\pi}{2} - 1 \approx 0.57$$

**Solution:** Here to do the integration we must use integration by parts. We will do the evaluation as we integrate. Let

$$\begin{aligned} u &= x & dv &= \cos(x) dx \\ du &= dx & v &= \sin(x). \end{aligned}$$

Then the integration by parts formula says that

$$\begin{aligned} \int_0^{\pi/2} x \cos(x) dx &= x \sin(x) \Big|_0^{\pi/2} - \int_0^{\pi/2} \sin(x) dx \\ &= \left( \frac{\pi}{2} \sin\left(\frac{\pi}{2}\right) - 0 \sin(0) \right) - \left[ -\cos(x) \right]_0^{\pi/2} \\ &= \frac{\pi}{2} - [-0 - (-1)] \\ &= \frac{\pi}{2} - 1 \approx 0.57 \end{aligned}$$

$$(h) \int_e^{e^2} \frac{1}{x \ln(x)} dx = \ln(2) \approx 0.69315$$

**Solution:** This problem requires the substitution  $u = \ln(x)$ . Notice that, from this,  $du = \frac{1}{x} dx$ , and so the *indefinite* integral becomes

$$\int \frac{1}{x \ln(x)} dx = \int (\ln(x))^{-1} \frac{1}{x} dx = \int u^{-1} du = \ln(|u|) + K = \ln(|\ln(x)|) + K.$$

We are actually looking for a *definite* integral, so we evaluate the simplest possible anti-derivative (the one with  $K = 0$ ):

$$\begin{aligned} \int_e^{e^2} \frac{1}{x \ln(x)} dx &= \ln(|\ln(x)|) \Big|_e^{e^2} = \ln(\ln(e^2)) - \ln(\ln(e)) \\ &= \ln(2) - \ln(1) = \ln(2) \approx 0.69315. \end{aligned}$$

$$(i) \int_0^{\ln(2)} x e^{2x} dx = 2 \ln(2) - \frac{3}{4} \approx 0.6363$$

**Solution:** This integral requires that we integrate by parts. Let

$$\begin{aligned} u &= x & dv &= e^{2x} dx \\ du &= dx & v &= \frac{1}{2} e^{2x}. \end{aligned}$$

Then we can compute the indefinite integral using the integration by parts formula:

$$\begin{aligned} \int x e^{2x} dx &= x \cdot \frac{1}{2} e^{2x} - \int \frac{1}{2} e^{2x} dx \\ &= \frac{1}{2} x e^{2x} - \frac{1}{2} \int e^{2x} dx \\ &= \frac{1}{2} x e^{2x} - \frac{1}{2} \cdot \frac{1}{2} e^{2x} + K \\ &= \frac{1}{2} x e^{2x} - \frac{1}{4} e^{2x} + K. \end{aligned}$$

Since we're actually interested in the definite integral, we can use any anti-derivative we wish. We choose the simplest ( $K = 0$ ) and compute:

$$\begin{aligned} \int_0^{\ln(2)} xe^{2x} dx &= \left( \frac{1}{2}xe^{2x} - \frac{1}{4}e^{2x} \right) \Big|_0^{\ln(2)} \\ &= \left( \frac{1}{2}\ln(2)e^{2\ln(2)} - \frac{1}{4}e^{2\ln(2)} \right) - \left( \frac{1}{2}(0)e^0 - \frac{1}{4}e^0 \right) \\ &= \left( 2\ln(2) - \frac{4}{4} \right) - \left( 0 - \frac{1}{4} \right) \\ &= 2\ln(2) - \frac{3}{4} \approx 0.6363. \end{aligned}$$

(The tricky thing here is that  $e^{2\ln(2)} = 4$ . This is because  $2\ln(2) = \ln(2^2) = \ln(4)$ , so  $e^{\ln(4)} = 4$ .)

3 Find the area bounded between the following functions:

$$f(x) = \sqrt{x-1} \quad g(x) = -\sqrt{x-1} \quad h(x) = 7-x.$$

**Solution:** Here the graph is shown in Figure 1. The points of intersection are found by setting the equations

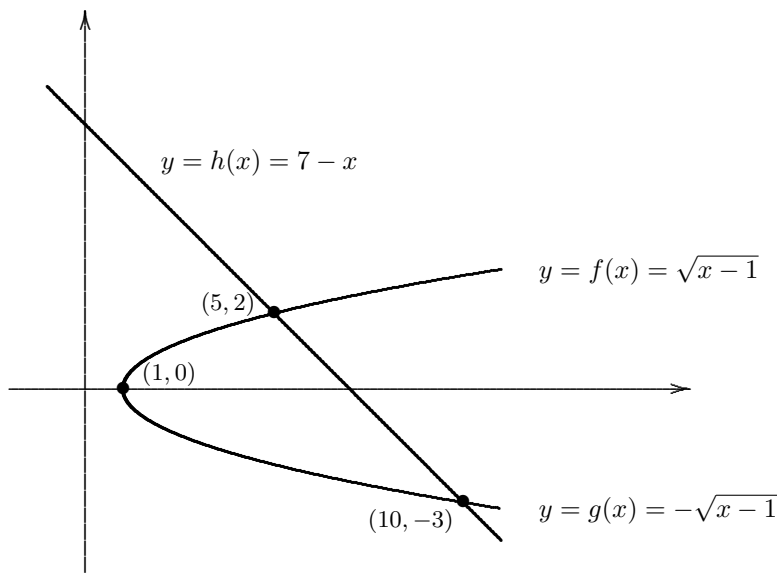


Figure 1: Graph of three curves for problem 3

of the curves equal to each other:

$$\pm\sqrt{x-1} = 7-x.$$

We solve for these points by squaring both sides, then putting everything on one side:

$$\begin{aligned} (\pm\sqrt{x-1})^2 &= (7-x)^2 \\ x-1 &= 49-14x+x^2 \\ 0 &= x^2-15x+50 \\ 0 &= (x-10)(x-5). \end{aligned}$$

Thus the curves intersect at  $(x, y) = (5, 2)$  and  $(x, y) = (10, -3)$ . (The two curves  $y = \pm\sqrt{x-1}$  also intersect at  $(x, y) = (1, 0)$ .)

The area bounded by these three curves therefore has  $y = -\sqrt{x-1}$  as the bottom curve for the entire region. The top curve is  $y = \sqrt{x-1}$  from  $x = 1$  to  $x = 5$ , and  $y = 7 - x$  from  $x = 5$  to  $x = 10$ . Thus the total area is given by

$$\begin{aligned} A &= \int_1^5 [\sqrt{x-1} - (-\sqrt{x-1})] dx + \int_5^{10} [(7-x) - (-\sqrt{x-1})] dx \\ &= 2 \int_1^5 (x-1)^{1/2} dx + \int_5^{10} [7-x + (x-1)^{1/2}] dx \\ &= 2 \left[ \frac{2}{3}(x-1)^{3/2} \right]_1^5 + \left[ 7x - \frac{x^2}{2} + \frac{2}{3}(x-1)^{3/2} \right]_5^{10} \\ &= 2 \left[ \frac{2}{3}(5-1)^{3/2} - 0 \right] + \left[ \left( 7(10) - \frac{10^2}{2} + \frac{2}{3}(10-1)^{3/2} \right) - \left( 7(5) - \frac{5^2}{2} + \frac{2}{3}(5-1)^{3/2} \right) \right] \\ &= \frac{32}{3} + \left[ 38 - \left( 35 - \frac{25}{2} + \frac{16}{3} \right) \right] = \frac{125}{6} \approx 20.8333 \end{aligned}$$

- 4 Scientists predict that if current global warming trends continue the North polar region's icebergs will melt, increasing the volume of water in the the oceans. Measurements indicate that  $f(t)$  billions of liters per year will flow into the oceans, where  $f(t)$  is given by

$$f(t) = e^{t/2} + 10t.$$

If they are correct, how many billion liters of water will be added to the oceans in the next five years ?

**Solution:** If the scientists are correct, then the amount of water that will flow into the ocean in the next five years is

$$\int_0^5 (e^{t/2} + 10t) dt \text{ billion liters.}$$

We can compute this definite integral to get a numerical answer:

$$\begin{aligned} \int_0^5 (e^{t/2} + 10t) dt &= \left[ 2e^{t/2} + 5t^2 \right]_0^5 \\ &= \left( 2e^{5/2} + 5(5)^2 \right) - \left( 2e^{0/2} + 5(0)^2 \right) \\ &= 2e^{5/2} + 123 \approx 147.35 \text{ billion liters.} \end{aligned}$$

- 5 Calculate the average value of the function  $f(x) = x \cos(ax)$  on the interval  $0 \leq x \leq \frac{2\pi}{a}$  for constant  $a > 0$ .

**Solution:** The average value of a function  $y = f(x)$  on the interval from  $x = a$  to  $x = b$  is given by

$$\text{A.V.} = \frac{1}{b-a} \int_a^b f(x) dx.$$

Here our endpoints are from  $x = 0$  to  $x = 2\pi/a$  (this  $a$  is unrelated to the endpoint  $a$  above), and our function is  $f(x) = x \cos(ax)$ . Thus the average value we want is

$$\text{A.V.} = \frac{1}{2\pi/a - 0} \int_0^{2\pi/a} x \cos(ax) dx = \frac{a}{2\pi} \int_0^{2\pi/a} x \cos(ax) dx.$$

If we make the substitution  $t = ax$ , so  $dt = a dx$ , we get

$$\text{A.V.} = \frac{a}{2\pi} \int_{t=0}^{t=2\pi} \frac{t}{a} \cos(t)a dt = \frac{a}{2\pi} \int_0^{2\pi} t \cos(t) dt.$$

This we can do by integration by parts: let  $u = t$  and  $dv = \cos(t) dt$ . Then  $du = dt$  and  $v = \sin(t)$ . Thus we can compute the indefinite integral

$$\int t \cos(t) dt = t \sin(t) - \int 1 \cdot \sin(t) dt = t \sin(t) + \cos(t) + K.$$

Thus the average value is given by

$$\begin{aligned} \text{A.V.} &= \frac{a}{2\pi} \int_0^{2\pi} t \cos(t) dt. = \frac{a}{2\pi} [t \sin(t) + \cos(t)]_0^{2\pi} \\ &= \frac{a}{2\pi} [(2\pi \sin(2\pi) + \cos(2\pi)) - (0 \sin(0) + \cos(0))] = 0. \end{aligned}$$

Thus the average value is 0.

6 Find the area bounded by the following functions:

$$f(x) = x^3 + x^2 - x + 10 \qquad g(x) = x^2 + 10$$

**Solution:** These two functions intersect when  $f(x) = g(x)$ , or

$$x^3 + x^2 - x + 10 = x^2 + 10.$$

This happens when  $x^3 - x = 0$ , or  $x(x-1)(x+1) = 0$ , or  $x = 0$  or  $x = \pm 1$ . By plotting points, we see that  $f(x) > g(x)$  when  $-1 < x < 0$ , and  $f(x) < g(x)$  when  $0 < x < +1$ . Thus the area bounded by these two functions is given by

$$\begin{aligned} \int_{-1}^0 [f(x) - g(x)] dx + \int_0^1 [g(x) - f(x)] dx &= \int_{-1}^0 (x^3 - x) dx + \int_0^1 (x - x^3) dx \\ &= \left[ \frac{x^4}{4} - \frac{x^2}{2} \right]_{-1}^0 + \left[ \frac{x^2}{2} - \frac{x^4}{4} \right]_0^1 \\ &= \left[ (0 - 0) - \left( \frac{(-1)^4}{4} - \frac{(-1)^2}{2} \right) \right] + \left[ \left( \frac{1^2}{2} - \frac{1^4}{4} \right) - (0 - 0) \right] \\ &= \frac{1}{4} + \frac{1}{4} = \frac{1}{2}. \end{aligned}$$

Thus the area is  $\frac{1}{2}$  square units.

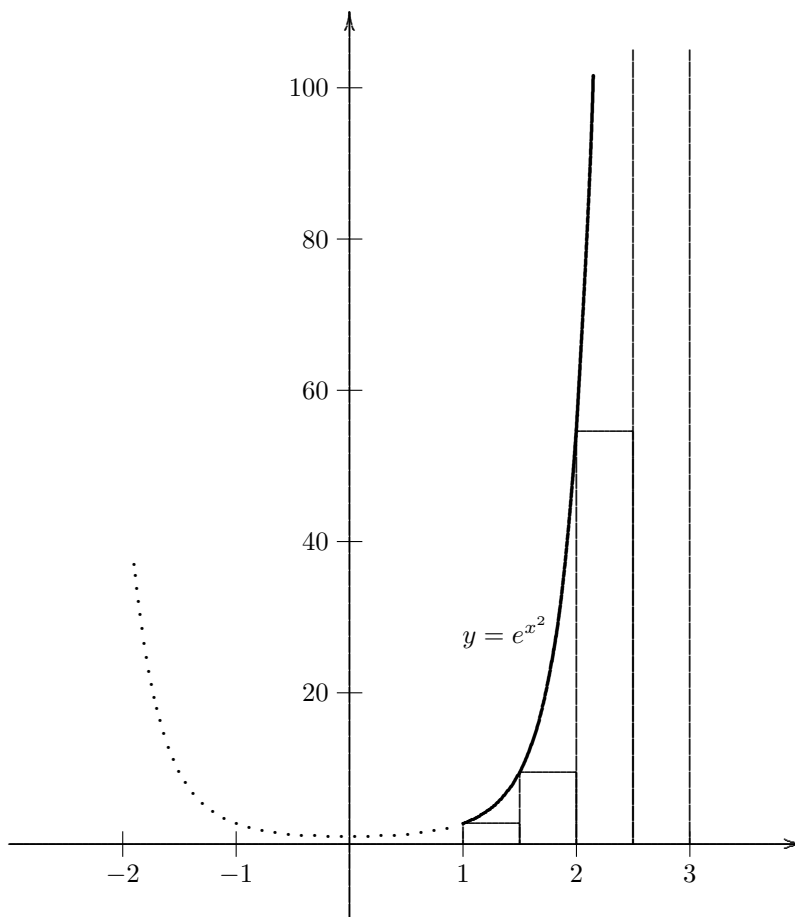
7 Approximate the integral  $\int_1^3 e^{x^2} dx$  by partitioning the interval  $[1, 3]$  into 4 subintervals and choosing  $u$  as the left endpoint of each subinterval. Round your answer to 4 digits after the decimal.

**Solution:** A picture of the graph of  $y = e^{x^2}$  with the appropriate boxes drawn in is shown in Figure 2. (This is missing a large chunk of the graph: when  $x = 3$ ,  $y = e^{x^2}$  is over 8,000. Even the rectangle for the interval  $[2.5, 3]$  has its top well out of view, since  $e^{(2.5)^2}$  is over 500.)

We begin by partitioning the interval  $[1, 3]$  into 4 subintervals of equal length  $\Delta x$ . (This length is, of course,  $\Delta x = \frac{b-a}{n} = \frac{3-1}{4} = 0.5$ .) Then, in each subinterval  $[x_{i-1}, x_i]$ , we choose  $u_i$  to be the left endpoint  $u_i = x_{i-1}$ . This means  $u_1 = 1$ ,  $u_2 = 1.5$ ,  $u_3 = 2$ , and  $u_4 = 2.5$ . Then the integral is approximately

$$\begin{aligned} \int_1^3 e^{x^2} dx &= f(u_1)\Delta x + f(u_2)\Delta x + f(u_3)\Delta x + f(u_4)\Delta x \\ &= f(1)\Delta x + f(1.5)\Delta x + f(2)\Delta x + f(2.5)\Delta x \\ &= e^{1^2}(0.5) + e^{(1.5)^2}(0.5) + e^{2^2}(0.5) + e^{(2.5)^2}(0.5) \\ &\approx 292.4085. \end{aligned}$$

(To four decimal places, the actual value of the definite integral is, according to Mathematica, 1443.0825. The picture should make it clear that we are underestimating the area significantly. If we'd used the right endpoint of each interval rather than the left, and thus *overestimated* the area, we'd get an estimate of 4342.5913 rather than 292.4085. This estimate is well above the true value, as expected.)

Figure 2: Graph of  $y = e^{x^2}$  (with boxes) for Problem 7

- 8 Find the function  $f(x)$  for which  $f'(x) = 2e^{-x} - \frac{3}{x+1}$  and  $f(0) = 2$ .

**Solution:** We know that, if we write

$$g(x) = \int \left( 2e^{-x} - \frac{3}{x+1} \right) dx,$$

then the difference  $f(x) - g(x)$  is constant. Put another way,  $f(x)$  is  $g(x)$  for some appropriate choice of integration constant. Thus the bulk of the work is finding the given integral. But this isn't difficult: using a substitution  $u = -x$  (so  $du = -dx$ ) on the first part of the integral, and a substitution  $v = x + 1$  (so  $dv = dx$ ) on the second part, we get that

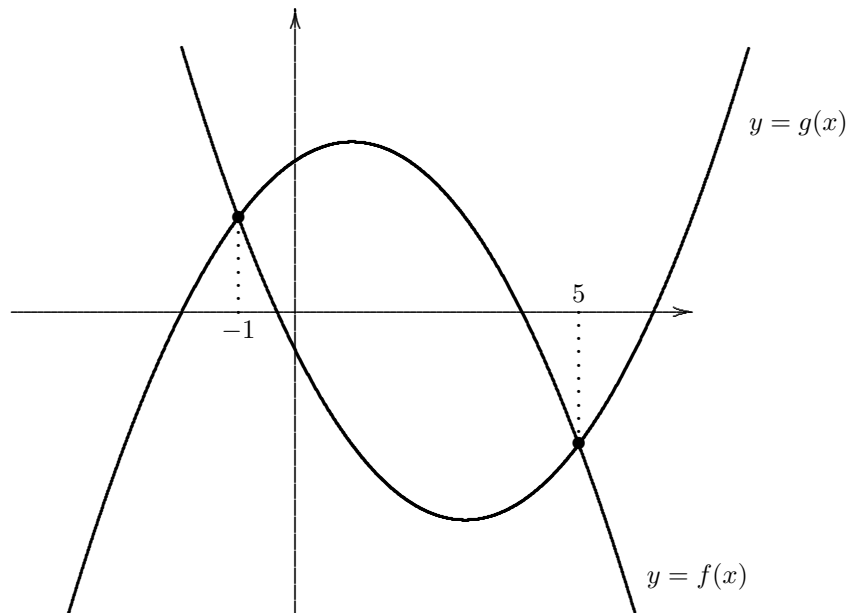
$$\int \left( 2e^{-x} - \frac{3}{x+1} \right) dx = -2e^{-x} - 3\ln(|x+1|) + K.$$

Thus  $f(x) = -2e^{-x} - 3\ln(|x+1|) + K$ , where we choose  $K$  so that  $f(0) = 2$ . Plugging in, we see that  $f(0) = -2e^0 - 3\ln(1) + K = -2 + 0 + K = K - 2$ . For this to equal 2 we must have  $K = 4$ . Hence  $f(x) = -2e^{-x} - 3\ln(|x+1|) + 4$ .

9 Find the area bounded by the following functions:

$$f(x) = -x^2 + 2x + 8 \quad g(x) = x^2 - 6x - 2.$$

**Solution:** The picture looks like this:



The points of intersection of the two curves are found by setting the two equations equal. We compute:

$$-x^2 + 2x + 8 = x^2 - 6x - 2$$

or

$$0 = 2x^2 - 8x - 10 = 2(x - 5)(x + 1).$$

Thus the two curves cross when  $x = -1$  (that is, at the point  $(x, y) = (-1, 5)$ ) and when  $x = 5$  (at the point  $(x, y) = (5, -7)$ ). The curve  $y = f(x)$  is the top boundary of the region, and  $y = g(x)$  is the bottom boundary. Thus the area bounded by the two curves is given by

$$\begin{aligned} \text{Area} &= \int_{-1}^5 [f(x) - g(x)] \, dx \\ &= \int_{-1}^5 [(-x^2 + 2x + 8) - (x^2 - 6x - 2)] \, dx \\ &= \int_{-1}^5 (-2x^2 + 8x + 10) \, dx \\ &= \left[ -\frac{2}{3}x^3 + 4x^2 + 10x \right]_{-1}^5 \\ &= -\frac{2}{3}[(5)^3 - (-1)^3] + 4[(5)^2 - (-1)^2] + 10[5 - (-1)] \\ &= 72. \end{aligned}$$

Hence the answer we want is 72 square units.

10 Suppose  $f'(x) = \frac{1}{\sqrt{x+4}} + 6e^{2x}$ . If  $f(0) = 3$ , find  $f(x)$ .

**Solution:** The idea here is that  $f(x)$  must be an anti-derivative of the given function  $f'(x) = \frac{1}{\sqrt{x+4}} + 6e^{2x}$ , so we begin by computing the general anti-derivative (that is, the indefinite integral):

$$\begin{aligned} \int \left( \frac{1}{\sqrt{x+4}} + 6e^{2x} \right) dx &= \int (x+4)^{-1/2} dx + 6 \int e^{2x} dx \\ &= \frac{1}{1/2}(x+4)^{1/2} + 6 \cdot \frac{1}{2}e^{2x} + K \\ &= 2\sqrt{x+4} + 3e^{2x} + K. \end{aligned}$$

Thus  $f(x) = 2\sqrt{x+4} + 3e^{2x} + K$ , where  $K$  is chosen so that  $f(0) = 3$ . We plug in  $x = 0$  and find that  $f(0) = 2\sqrt{4} + 3e^0 + K = 4 + 3 + K = 7 + K$ , so in order for this to equal 3, we must have  $K = -4$ . Thus  $f(x) = 2\sqrt{x+4} + 3e^{2x} - 4$ .

11 Calculate the average value of the function  $f(x) = x\sqrt{x+1}$  on the interval  $0 \leq x \leq 9$ .

**Solution:** The average value A.V. of a function  $y = f(x)$  on an interval  $a \leq x \leq b$  is given by the equation

$$\text{A.V.} = \frac{1}{b-a} \int_a^b f(x) dx.$$

We use this equation with  $f(x) = x\sqrt{x+1}$ ,  $a = 0$ , and  $b = 9$ ; we get

$$\text{A.V.} = \frac{1}{9-0} \int_0^9 x\sqrt{x+1} dx.$$

One way to compute this integral is via integration by parts. We will use

$$\begin{aligned} u &= x & dv &= (x+1)^{1/2} dx \\ du &= dx & v &= \frac{2}{3}(x+1)^{3/2}. \end{aligned}$$

Thus (just computing the indefinite integral)

$$\begin{aligned} \int x\sqrt{x+1} dx &= \frac{2}{3}x(x+1)^{3/2} - \frac{2}{3} \int (x+1)^{3/2} dx \\ &= \frac{2}{3}x(x+1)^{3/2} - \frac{2}{3} \cdot \frac{2}{5}(x+1)^{5/2} + K. \end{aligned}$$

From this we may evaluate the definite integral

$$\begin{aligned} \text{A.V.} &= \frac{1}{9} \int_0^9 x\sqrt{x+1} dx \\ &= \frac{1}{9} \left[ \frac{2}{3}x(x+1)^{3/2} - \frac{2}{3} \cdot \frac{2}{5}(x+1)^{5/2} \right]_0^9 \\ &= \frac{1}{9} \left[ \left( \frac{2}{3} \cdot 9(10)^{3/2} - \frac{4}{15}(10)^{5/2} \right) - \left( \frac{2}{3} \cdot 0(1)^{3/2} - \frac{4}{15}(1)^{5/2} \right) \right] \\ &= \frac{1}{9} \left[ 60\sqrt{10} - \frac{80}{3}\sqrt{10} + \frac{4}{15} \right] \\ &= \frac{500\sqrt{10} + 4}{135} \approx 11.7418. \end{aligned}$$