1 (a) What system of equations is represented by the following augmented matrix?

$$\left[\begin{array}{cccc|c} 2 & -2 & 6 & 0 & -18 \\ 0 & 1 & -1 & 0 & 7 \\ 0 & -1 & 1 & 2 & -9 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

Solution: The matrix corresponds to the following system of equations:

$$\begin{aligned} 2x_1 - 2x_2 + 6x_3 + 0x_4 &= -18 \\ 0x_1 + 1x_2 - 1x_3 + 0x_4 &= 7 \\ 0x_1 - 1x_2 + 1x_3 + 2x_4 &= -9 \\ 0x_1 + 0x_2 + 0x_3 + 0x_4 &= 0 \end{aligned}$$

or, written in a more readable format,

$$\begin{aligned} 2x_1 - 2x_2 + 6x_3 \quad\quad &= -18 \\ x_2 - x_3 \quad\quad &= 7 \\ -x_2 + x_3 + 2x_4 &= -9 \\ 0 &= 0 \end{aligned}$$

(b) Put the above matrix into *reduced* row echelon form. Show the details of your computation, not just the final answer.

Solution: We'll do our row reduction more or less without comment, simply noting the row operations at the side of each matrix:

$$\left[\begin{array}{cccc|c} 2 & -2 & 6 & 0 & -18 \\ 0 & 1 & -1 & 0 & 7 \\ 0 & -1 & 1 & 2 & -9 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] \longrightarrow \left[\begin{array}{cccc|c} 1 & -1 & 3 & 0 & -9 \\ 0 & 1 & -1 & 0 & 7 \\ 0 & -1 & 1 & 2 & -9 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] \quad R_1 = \tfrac{1}{2}r_1$$

$$\longrightarrow \left[\begin{array}{cccc|c} 1 & -1 & 3 & 0 & -9 \\ 0 & 1 & -1 & 0 & 7 \\ 0 & 0 & 0 & 2 & -2 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] \quad R_3 = r_3 + r_2$$

$$\longrightarrow \left[\begin{array}{cccc|c} 1 & 0 & 2 & 0 & -2 \\ 0 & 1 & -1 & 0 & 7 \\ 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] \quad \begin{array}{l} R_1 = r_1 + r_2 \\ \\ R_3 = \tfrac{1}{2}r_3 \end{array}$$

This is in reduced row echelon form.

(c) What is the solution of the linear system from part (a)?

Solution: The last matrix from part (b) corresponds to the linear system

$$\begin{aligned} x_1 \quad + 2x_3 \quad &= -2 \\ x_2 - \quad x_3 \quad &= \ \ 7 \\ x_4 &= -1 \\ 0 &= \ \ 0 \end{aligned}$$

which gives us the solution $x_1 = -2 - 2x_3$, $x_2 = 7 + x_3$, $x_4 = -1$, and x_3 is arbitrary, or $(x_1, x_2, x_3, x_4) = (-2 - 2x_3, 7 + x_3, x_3, -1)$.

2 (a) Suppose $A = \begin{bmatrix} 0 & -1 & 1 \\ 1 & 7 & -5 \\ 0 & 3 & -2 \end{bmatrix}$. Find a, b, and c so that $B = \begin{bmatrix} 1 & 1 & a \\ 2 & b & 1 \\ c & 0 & 1 \end{bmatrix}$ is the inverse of A. You must show your work to justify your answer!

Hint: Don't simply calculate A^{-1}! What properties does A^{-1} have?

Solution: The simplest thing to do in this case is *not* to calculate A^{-1}, but rather simply to compute A times B and compare it to the identity I. Since B is the inverse of A, we should have $AB = I$:

$$AB = \begin{bmatrix} 0 & -1 & 1 \\ 1 & 7 & -5 \\ 0 & 3 & -2 \end{bmatrix} \begin{bmatrix} 1 & 1 & a \\ 2 & b & 1 \\ c & 0 & 1 \end{bmatrix} = \begin{bmatrix} c-2 & -b & 0 \\ 15-5c & 7b+1 & a+2 \\ 6-2c & 3b & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = I.$$

Comparing these last two matrices, we see that $c = 3$, $b = 0$, and $a = -2$. From this we see that

$$A^{-1} = \begin{bmatrix} 1 & 1 & -2 \\ 2 & 0 & 1 \\ 3 & 0 & 1 \end{bmatrix}.$$

(b) Solve the following linear system of equations:

$$\begin{aligned} -y + \ \ z &= \ \ 2 \\ x + 7y - 5z &= -3 \\ 3y - 2z &= \ \ 1 \end{aligned}$$

Hint: Why is this part (b)?

Solution: To answer the hint, let us write this system in matrix form:

$$\begin{bmatrix} 0 & -1 & 1 \\ 1 & 7 & -5 \\ 0 & 3 & -2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2 \\ -3 \\ 1 \end{bmatrix}.$$

If we think of this as $AX = C$ (and I'm using C instead of B only because we had a matrix named B in part (a)), then this A is precisely the matrix in part (a)! Thus it will be straightforward to use matrix inversion to solve this system: $X = A^{-1}C$, so

$$X = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = A^{-1}C = \begin{bmatrix} 1 & 1 & -2 \\ 2 & 0 & 1 \\ 3 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ -3 \\ 1 \end{bmatrix} = \begin{bmatrix} 1(2)+1(-3)-2(1) \\ 2(2)+0(-3)+1(1) \\ 3(2)+0(-3)+1(1) \end{bmatrix} = \begin{bmatrix} -3 \\ 5 \\ 7 \end{bmatrix}.$$

Written another way, we've found that $(x, y, z) = (-3, 5, 7)$.

3 (a) While waiting in line at a popular brunch spot, I watch three tables. One group gets 2 orders of eggs and 2 orders of waffles, plus 3 coffees; their bill is \$27.95. Another table has 3 orders of waffles, 1 of eggs, and 4 coffees; their bill is \$28.85. The third table has 4 orders of eggs and 2 coffees; this comes to \$27.90.

Write down a linear system that we could solve to find the prices of eggs, waffles, and coffee. Clearly indicate which variable corresponds to each price. Please **DO NOT SOLVE** this system.

Solution: Let w be the price of an order of waffles, e the price of an order of eggs, and c the price of a coffee. Then the sentence

> "One group gets 2 orders of eggs and 2 orders of waffles, plus 3 coffees; their bill is \$27.95."

means that $2e + 2w + 3c = 27.95$. Similarly we can find two more equations, to get the system

$$\begin{aligned} 2e + 2w + 3c &= 27.95 \\ 1e + 3w + 4c &= 28.85 \\ 4e + 0w + 2c &= 27.90 \end{aligned}$$

This is all that was required.

This system, by the way, has a unique solution. In fact, and order of eggs costs $e = \$6.10$, an order of waffles costs $w = \$5.25$, and coffee costs $c = \$1.75$. You weren't, however, asked to find these prices.

(b) Suppose we have a 100 gallon tank full of water. There are 5 pounds of salt dissolved in the water. Salt water with concentration 0.01 pounds per gallon is being added at a rate of 2 gallons per minute and the (well-stirred) mixture is being poured out at the same rate.

Let $Q(t)$ be the quantity of salt (in pounds) in the tank at time t. Set up an initial value problem that is satisfied by $Q(t)$ and its derivative. **You do not need to solve this equation!**

Solution: We're told that $Q(t)$ be the quantity (in pounds) of salt in the vat as a function of the time t (in minutes). Then the rate of change of salt in the vat is

$$\text{rate of change} = +\Big(\text{ rate in }\Big)\Big(\text{ concentration in }\Big) - \Big(\text{ rate out }\Big)\Big(\text{ concentration out }\Big).$$

Written another way, this equation is

$$Q'(t) = +\left(2\ \frac{\text{gals}}{\text{min}}\right)\left(0.01\ \frac{\text{lbs}}{\text{gal}}\right) - \left(2\ \frac{\text{gals}}{\text{min}}\right)\left(\frac{Q(t)\ \text{lbs}}{100\ \text{gals}}\right).$$

The only really tricky part here is noting that the concentration of the liquid flowing out is that of the mixture, which is the quantity $Q(t)$ of salt over the volume of liquid (100 gallons).

Thus our differential equation is $Q'(t) = 0.02 - \frac{2}{100}Q(t)$. The initial condition given is that at $t = 0$ there are 5 pounds of salt, so $Q(0) = 5$. Thus our initial value problem is

$$\begin{aligned} Q'(t) &= 0.02 - 0.02Q(t) \\ Q(0) &= 5. \end{aligned}$$

We're not asked to solve this, but this has solution $Q(t) = 1 + 4e^{-0.02t}$. Thus, in the limit (as t grows without bound), there will be 1 pound of salt in the water. Since there are 100 gallons of liquid, the limiting concentration is $\frac{1 \text{ pound}}{100 \text{ gallons}} = 0.01$ pounds per gallon, which is the concentration of the salt water flowing in.

4 Find the general solution of the following differential equations:

(a) $y' = \dfrac{x^2+1}{y^2}$

Solution: This is a separable differential equation. We re-write it as

$$\frac{dy}{dx} = \frac{x^2+1}{y^2} \qquad \text{or} \qquad y^2\,dy = \left(x^2+1\right)\,dx.$$

Integrating both sides of this gives

$$\int y^2\,dy = \int \left(x^2+1\right)\,dx \qquad \text{or} \qquad \frac{1}{3}y^3 = \frac{1}{3}x^3 + x + K.$$

We solve for y by multiplying by 3 and taking the cube root:

$$y = \sqrt[3]{x^3+3x+3K} = \sqrt[3]{x^3+3x+C},$$

where $C = 3K$ is an unknown constant.

(b) $y'' - 4y' + 13y = 0$

Solution: This is a second order linear homogeneous differential equation with constant coefficients, so we look at the characteristic equation

$$r^2 - 4r + 13 = 0.$$

This has roots

$$r = \frac{+4 \pm \sqrt{(-4)^2 - 4(1)(13)}}{2(1)} = \frac{4 \pm \sqrt{16-52}}{2} = \frac{4 \pm \sqrt{-36}}{2} = \frac{4 \pm 6i}{2} = 2 \pm 3i.$$

Since the roots aren't real, we're in the third case from the class notes. That is, the general solution is

$$y(t) = e^{2t}\Big(P\cos(3t) + Q\sin(3t)\Big),$$

where P and Q are constants.

5 Find the solution of the following initial value problems:

(a) $xy' = x^3 + 2y$ where $y(1) = 5$

Solution: We begin by putting the equation into standard form for first order linear differential equations. This involves subtracting $2y$ from both sides and dividing the entire equation by x:

$$y' - \frac{2}{x}y = x^2.$$

Thus $P(x) = -2/x$ and $G(x) = x^2$.

We solve this kind of equation by multiplying by an integrating factor. This is $h(x) = e^{\int P(x)\,dx}$. Since

$$\int P(x)\,dx = \int -\frac{2}{x}\,dx = -2\int \frac{1}{x}\,dx = -2\ln(x) = \ln\left(x^{-2}\right).$$

(We've omitted the arbitrary constant "$+K$" as we're interested in *one* integrating factor, not *all possible* integrating factors.) The integrating factor is thus

$$h(x) = e^{\int P(x)\,dx} = e^{\ln(x^{-2})} = x^{-2} = \frac{1}{x^2}.$$

Now (I hope) you see the idea behind writing $-2\ln(x)$ as $\ln(x^{-2})$: this makes it easy to simplify $e^{\int P(x)\,dx}$.

We multiply the equation (in standard form) by $h(x)$ to get

$$\frac{1}{x^2}\left(y' - \frac{2}{x}y = x^2\right) \qquad \text{or} \qquad \frac{1}{x^2}y' - \frac{2}{x^3}y = \frac{x^2}{x^2} = 1.$$

The way we've chosen our integrating factor (the whole point behind it, really) is that the left-hand side is now a derivative:

$$\left(\frac{1}{x^2}y\right)' = 1.$$

Integrating both sides, we get

$$\frac{1}{x^2}y = \int 1\,dx = x + K,$$

or (solving) $y = x^2(x + K) = x^3 + Kx^2$.

Plugging the initial condition $y(1) = 5$ into this gives us $5 = 1^3 + K(1)^2 = 1 + K$, so $K = 4$. Thus our final answer is $y = x^3 + 4x^2$.

(b) $y'' - 4y' - 12y = 0$ where $y(0) = 1$ and $y'(0) = 4$

Solution: This is a second order linear homogeneous differential equation with constant coefficients, so we look at the characteristic equation

$$r^2 - 4r - 12 = 0.$$

This factors into $(r + 2)(r - 6) = 0$, so $r_1 = -2$ and $r_2 = 6$. We could also use the quadratic equation:

$$r = \frac{+4 \pm \sqrt{(-4)^2 - 4(1)(-12)}}{2(1)} = \frac{4 \pm \sqrt{16 + 48}}{2} = \frac{4 \pm \sqrt{64}}{2} = \frac{4 \pm 8}{2} = -2,\ 6.$$

Since the roots are real and distinct, we're in the first case from the class notes. That is, the general solution is

$$y(t) = Pe^{-2t} + Qe^{6t},$$

where P and Q are constants. We solve for these constants using the initial conditions. The condition $y(0) = 1$ means that

$$1 = Pe^{-2(0)} + Qe^{6(0)} \qquad \text{or} \qquad 1 = P + Q.$$

The second initial condition requires we differentiate first: $y' = -2Pe^{-2t} + 6Qe^{6t}$. Thus $y'(0) = 4$ means that

$$4 = -2Pe^{-2(0)} + 6Qe^{6(0)} \qquad \text{or} \qquad 4 = -2P + 6Q.$$

Solving this system of two equations gives $P = 1/4$ and $Q = 3/4$. Thus our final answer is

$$y(t) = \frac{1}{4}e^{-2t} + \frac{3}{4}e^{6t}.$$