1 (a) Consider the matrix $A = \begin{bmatrix} 1 & 3 & 7 \\ 4 & 1 & 2 \\ 3 & 1 & 2 \end{bmatrix}$. One of the following matrices is actually the inverse matrix A^{-1}. Decide which matrix is actually A^{-1}. Justify your answer with some computations.

$$B = \begin{bmatrix} 5 & -1 & 0 \\ 1 & -2 & 11 \\ -1 & -6 & 2 \end{bmatrix} \qquad C = \begin{bmatrix} 4 & -1 & 0 \\ 1 & 2 & 1 \\ -1 & 3 & -12 \end{bmatrix} \qquad D = \begin{bmatrix} 0 & 1 & -1 \\ -2 & -19 & 26 \\ 1 & 8 & -11 \end{bmatrix}.$$

Solution: One simple, direct way to find the inverse is to compute the products AB, AC, and AD. We do this here.

The entry in the first row and first column of AB is

$$(AB)_{1,1} == \begin{bmatrix} 1 & 3 & 7 \end{bmatrix} \begin{bmatrix} 5 \\ 1 \\ -1 \end{bmatrix} = (1)(5) + (3)(1) + (7)(-1) = 1.$$

This is the same as the $(1,1)$ entry in $I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$, so we have to continue. The entry $(AB)_{1,2}$ from the first row, second column of AB is

$$(AB)_{1,2} == \begin{bmatrix} 1 & 3 & 7 \end{bmatrix} \begin{bmatrix} -1 \\ -2 \\ -6 \end{bmatrix} = (1)(-1) + (3)(-2) + (7)(-6) = -49.$$

Similarly, if you prefer, the entry $(AB)_{2,1}$ from the second row, first column of AB is

$$(AB)_{2,1} == \begin{bmatrix} 4 & 1 & 2 \end{bmatrix} \begin{bmatrix} 5 \\ 1 \\ -1 \end{bmatrix} = (4)(5) + (1)(1) + (2)(-1) = 19.$$

In both cases, this entry does *not* agree with the corresponding entry in I, so $AB \neq I$.

Similarly, we can compute $(AC)_{1,1}$, the entry in the first row, first column of AC. This entry is

$$(AC)_{1,1} = \begin{bmatrix} 1 & 3 & 7 \end{bmatrix} \begin{bmatrix} 4 \\ 1 \\ -1 \end{bmatrix} = (1)(4) + (3)(1) + (7)(-1) = 0.$$

Since the $(1,1)$ entry of I is 1, this shows that $AC \neq I$. This leaves us with D, and we can compute AD and see that we get I:

$$AD = \begin{bmatrix} 1 & 3 & 7 \\ 4 & 1 & 2 \\ 3 & 1 & 2 \end{bmatrix} \begin{bmatrix} 0 & 1 & -1 \\ -2 & -19 & 26 \\ 1 & 8 & -11 \end{bmatrix} = \begin{bmatrix} 0-6+7 & 1-57+56 & -1+78-77 \\ 0-2+2 & 4-19+16 & -4+26-22 \\ 0-2+2 & 3-19+16 & -3+26-22 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Thus D must be A^{-1}, as it produces the identity matrix I when multiplied by A.

Notice that we only had to compute one or two entries of AB and AC before we could tell that they couldn't possibly be I. We could have multiplied them out completely, but that would have been much more work. Being a bit obsessive, however, we compute AB and AC to show that they are not, in fact, I:

$$AB = \begin{bmatrix} 1 & 3 & 7 \\ 4 & 1 & 2 \\ 3 & 1 & 2 \end{bmatrix} \begin{bmatrix} 5 & -1 & 0 \\ 1 & -2 & 11 \\ -1 & -6 & 2 \end{bmatrix} = \begin{bmatrix} 1 & -49 & 47 \\ 19 & -18 & 15 \\ 14 & -17 & 15 \end{bmatrix} \neq \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

and

$$AC = \begin{bmatrix} 1 & 3 & 7 \\ 4 & 1 & 2 \\ 3 & 1 & 2 \end{bmatrix} \begin{bmatrix} 4 & -1 & 0 \\ 1 & 2 & 1 \\ -1 & 3 & -12 \end{bmatrix} = \begin{bmatrix} 0 & 26 & -81 \\ 15 & 4 & -23 \\ 11 & 5 & -23 \end{bmatrix} \neq \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Thus neither B nor C is A^{-1}.

Alternatively, we can find A^{-1} by row reducing $[\,A\,|\,I\,]$ to $[\,I\,|\,A^{-1}\,]$. We can then identify A^{-1} as one of the listed matrices. We do this now:

$$[\,A\,|\,I\,] = B = \left[\begin{array}{ccc|ccc} 1 & 3 & 7 & 1 & 0 & 0 \\ 4 & 1 & 2 & 0 & 1 & 0 \\ 3 & 1 & 2 & 0 & 0 & 1 \end{array}\right] \longrightarrow \left[\begin{array}{ccc|ccc} 1 & 3 & 7 & 1 & 0 & 0 \\ 0 & -11 & -26 & -4 & 1 & 0 \\ 0 & -8 & -19 & -3 & 0 & 1 \end{array}\right] \quad \begin{array}{l} R_2 = r_2 - 4r_1 \\ R_3 = r_3 - 3r_1 \end{array}$$

$$\longrightarrow \left[\begin{array}{ccc|ccc} 1 & 3 & 7 & 1 & 0 & 0 \\ 0 & 1 & 26/11 & 4/11 & -1/11 & 0 \\ 0 & -8 & -19 & -3 & 0 & 1 \end{array}\right] \quad R_2 = -\tfrac{1}{11}r_2$$

$$\longrightarrow \left[\begin{array}{ccc|ccc} 1 & 3 & 7 & 1 & 0 & 0 \\ 0 & 1 & 26/11 & 4/11 & -1/11 & 0 \\ 0 & 0 & -1/11 & -1/11 & -8/11 & 1 \end{array}\right] \quad R_3 = r_3 + 8r_2$$

$$\longrightarrow \left[\begin{array}{ccc|ccc} 1 & 3 & 7 & 1 & 0 & 0 \\ 0 & 1 & 26/11 & 4/11 & -1/11 & 0 \\ 0 & 0 & 1 & 1 & 8 & -11 \end{array}\right] \quad R_3 = -11r_3$$

$$\longrightarrow \left[\begin{array}{ccc|ccc} 1 & 3 & 0 & -6 & -56 & 77 \\ 0 & 1 & 0 & -2 & -19 & 26 \\ 0 & 0 & 1 & 1 & 8 & -11 \end{array}\right] \quad \begin{array}{l} R_1 = r_1 - 7r_3 \\ R_2 = r_2 - \tfrac{26}{11}r_3 \end{array}$$

$$\longrightarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & 1 & -1 \\ 0 & 1 & 0 & -2 & -19 & 26 \\ 0 & 0 & 1 & 1 & 8 & -11 \end{array}\right] \quad R_1 = r_1 - 3r_2 \quad .$$

This is $[\,I\,|\,A^{-1}\,]$, so $A^{-1} = D$

(b) Solve the following linear system of equations:

$$\begin{aligned} x + 3y + 7z &= 1 \\ 4x + y + 2z &= -1 \\ 3x + y + 2z &= 2 \end{aligned}$$

Solution: If we write this system in matrix format, we get

$$\begin{bmatrix} 1 & 3 & 7 \\ 4 & 1 & 2 \\ 3 & 1 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix},$$

or $AX = B$ with the matrix A from part (a)! Thus $X = A^{-1}B$ or

$$X = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = A^{-1}B = \begin{bmatrix} 0 & 1 & -1 \\ -2 & -19 & 26 \\ 1 & 8 & -11 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix} = \begin{bmatrix} -3 \\ 69 \\ -29 \end{bmatrix}.$$

That is, the solution to the given system is $(x, y, z) = (-3, 69, -29)$.

2 (a) What system of equations is represented by the following augmented matrix?

$$\left[\begin{array}{cccc|c} 1 & 1 & 3 & 1 & 0 \\ 0 & 1 & 0 & 1 & -1 \\ 0 & 0 & 1 & 0 & 1/2 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

Solution: The matrix corresponds to the following system of equations:

$$\begin{aligned} 1x_1 + 1x_2 + 3x_3 + 1x_4 &= 0 \\ 0x_1 + 1x_2 + 0x_3 + 1x_4 &= -1 \\ 0x_1 + 0x_2 + 1x_3 + 0x_4 &= 1/2 \\ 0x_1 + 0x_2 + 0x_3 + 0x_4 &= 0 \end{aligned}$$

or, written in a more readable format,

$$\begin{aligned} x_1 + x_2 + 3x_3 + x_4 &= 0 \\ x_2 \qquad\quad + x_4 &= -1 \\ x_3 \qquad &= \tfrac{1}{2} \\ 0 &= 0 \end{aligned}$$

(b) Put the above matrix into *reduced* row echelon form. Show the details of your computation, not just the final answer.

Solution: The matrix in part (a) is in row echelon form. To put it in reduced row echelon form, we must make the terms above the "pivots" (the first non-zero entries in each row) zero. To do this, we will subtract one copy of the second row and three copies of the third row from the first. Thus we get the following:

$$\left[\begin{array}{cccc|c} 1 & 1 & 3 & 1 & 0 \\ 0 & 1 & 0 & 1 & -1 \\ 0 & 0 & 1 & 0 & 1/2 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] \longrightarrow \left[\begin{array}{cccc|c} 1 & 0 & 3 & 0 & 1 \\ 0 & 1 & 0 & 1 & -1 \\ 0 & 0 & 1 & 0 & 1/2 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] \qquad R_1 = r_1 - r_2$$

$$\longrightarrow \left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & -1/2 \\ 0 & 1 & 0 & 1 & -1 \\ 0 & 0 & 1 & 0 & 1/2 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] \qquad R_1 = r_1 - 3r_3.$$

This is the *reduced* row echelon form of the matrix from part (a).

(c) What is the solution of the linear system from part (a)?

Solution: We can now read off the solution from the reduced row echelon form. As in part (a), we re-write the matrix as a linear system to get

$$\begin{aligned} x_1 &= -\frac{1}{2} \\ x_2 + x_4 &= -1 \\ x_3 &= \frac{1}{2} \\ 0 &= 0. \end{aligned}$$

This shows that the solution to our system is

$$\begin{cases} x_1 &= -\frac{1}{2} \\ x_2 &= -1 - x_4 \\ x_3 &= \frac{1}{2} \end{cases}$$

where x_4 is a parameter (and is arbitrary).

3 (a) Find the solution of $y' - \dfrac{1}{x}\, y = x^2$ passing through the point $(x, y) = (1, 1)$.

Solution: This is a first-order linear differential equation that is already in the standard form $y' + P(x)y = G(x)$. Thus $P(x) = -\frac{1}{x}$, and so the integrating factor is

$$h(x) = e^{\int P(x)\,dx} = e^{\int -\frac{1}{x}\,dx} = e^{-\ln(x)} = e^{\ln(x^{-1})} = x^{-1} = \frac{1}{x}.$$

(The big trick to notice here is that we put the coefficient -1 of $\ln(x)$ into the exponent, so that we may use the fact that e^x and $\ln(x)$ are inverse functions.) Next we multiply through by the integrating factor $h(x)$ to get

$$\frac{1}{x}\, y' - \frac{1}{x^2}\, y = \frac{1}{x}\, x^2 \qquad \text{or} \qquad \left(\frac{1}{x}\, y\right)' = x.$$

(This is the other big trick – $h(x)$ is chosen so that the left-hand side *always* simplifies to the derivative of $h \cdot y$.) Integrating both sides of the equation on the right gives us

$$\frac{1}{x}\, y = \frac{1}{2}x^2 + K,$$

or $y = \frac{1}{2}x^3 + Kx$. Since $y = 1$ when $x = 1$, we must have $1 = \frac{1}{2}(1)^3 + K \cdot 1$, or $K = \frac{1}{2}$. Thus our final answer is $y = \frac{1}{2}x^3 + \frac{1}{2}x$, or $y = \frac{x^3+x}{2}$.

(b) Find the general solution of $x^2y' + xy = 9y'$.

Solution: This differential equation is both first-order linear and separable, so we can solve it using either technique. For completeness, we do so.

As a separable equation: First let's group the y' terms and write y' as $\frac{dy}{dx}$:

$$(x^2 - 9)y' + xy = 0 \qquad \text{or} \qquad (x^2 - 9)\frac{dy}{dx} = -xy.$$

Now we bring the y's to one side and the x's to the other by multiplying this equation by $\frac{dx}{(x^2-9)y}$:

$$\frac{dx}{y(x^2-9)}\left((x^2-9)\frac{dy}{dx} = -xy\right) \qquad \text{or} \qquad \frac{dy}{y} = -\frac{x\,dx}{x^2-9}.$$

Integrating both sides of this last equation (using the substitution $u = x^2 - 9$, so $du = 2x\,dx$ for the right-hand side) gives us

$$\ln(|y|) = -\frac{1}{2}\ln(|x^2-9|) + K.$$

Using the same trick as in part (a) to move the exponent up (that is, so that $-\frac{1}{2}\ln(|x^2-9|) = \ln(|x^2-9|^{-1/2})$), we can exponentiate both sides:

$$e^{\ln(|y|)} = e^{\ln(|x^2-9|^{-1/2})+K} = e^{\ln(|x^2-9|^{-1/2})}e^K \qquad \text{or} \qquad y = C|x^2-9|^{-1/2} = \frac{C}{\sqrt{|x^2-9|}}.$$

This is our general solution: $y = \dfrac{C}{\sqrt{|x^2-9|}}$.

As a first-order linear equation: To solve the differential equation as a first-order linear equation, we must begin by writing this in the standard form $y' + P(x)y = G(x)$:

$$(x^2-9)y' + xy = 0 \qquad \text{or} \qquad y' + \frac{x}{x^2-9}\,y = 0.$$

From this we get that $P(x) = \frac{x}{x^2-9}$ and $G(x) = 0$. Then the integrating factor is

$$h(x) = e^{\int P(x)\,dx} = e^{\int \frac{x}{x^2-9}\,dx} = e^{\frac{1}{2}\ln(|x^2-9|)} = e^{\ln(|x^2-9|^{1/2})} = |x^2-9|^{1/2} = \sqrt{|x^2-9|}.$$

We multiply the equation by this integrating factor to get

$$\sqrt{|x^2-9|}\left(y' + \frac{x}{x^2-9}\,y = 0\right) \qquad \text{or} \qquad \left(\sqrt{|x^2-9|}y\right)' = 0.$$

Integrate this to get

$$\left(\sqrt{|x^2-9|}y\right) = K \qquad \text{or} \qquad y = \frac{K}{\sqrt{|x^2-9|}},$$

as before.

4 Twenty (20) pounds of salt are dissolved in a tank containing 50 gallons of water. The water is constantly stirred, distributing the salt evenly throughout the water. Fresh water is pumped into the tank at a rate of 2 gallons per minute, and the well-mixed solution is pumped out at the same rate.

(a) Find an expression for the amount of salt after t minutes.

Solution: Let $A(t)$ be the amount (in pounds) of salt in the vat as a function of the time t (in minutes). Then the rate of change of salt in the vat is

$$\text{rate of change} = +\Big(\text{ rate in }\Big)\Big(\text{ concentration in }\Big) - \Big(\text{ rate out }\Big)\Big(\text{ concentration out }\Big).$$

Written another way, this equation is

$$A'(t) = +\left(2\ \frac{\text{gals}}{\text{min}}\right)\left(0\ \frac{\text{lbs}}{\text{gal}}\right) - \left(2\ \frac{\text{gals}}{\text{min}}\right)\left(\frac{A(t)\ \text{lbs}}{50\ \text{gals}}\right)$$

or $A'(t) = -0.04A(t)$. This is our differential equation, and our initial condition is $A(0) = 20$ pounds (the tank has 20 pounds of salt in it when the water begins to flow). This differential equation is, again, both a separable equation and first-order linear differential equation. We will, of course, solve it both ways.

As a separable equation: To separate the differential equation, we re-write $A'(t)$ as $\frac{dA}{dt}$. Thus the differential equation becomes

$$\frac{dA}{dt} = -0.04A \qquad \text{or} \qquad \frac{dA}{A} = -0.04\ dt.$$

Integrating this, we get $\ln(|A|) = -0.04t + K$. Exponentiating this, we get

$$e^{\ln(|A|)} = e^{-0.04t+K} = e^{-0.04t} \cdot e^K \qquad \text{or} \qquad A(t) = Ce^{-0.04t}.$$

The initial condition ($A = 20$ when $t = 0$) lets us find that $C = 20$. Thus the amount of salt after t minutes is $A(t) = 20e^{-0.04t}$ pounds.

As a first-order linear differential equation: We could write our differential equation in the standard form for first-order linear differential equations:

$$y' + P(x)y = G(x) \qquad \text{or, in this case,} \qquad A' + 0.04A = 0.$$

The integrating factor is $h(t) = e^{\int P(t)\ dt} = e^{\int 0.04\ dt} = e^{0.04t}$. When we multiply our equation by this factor, we get

$$e^{0.04t}\Big(A' + 0.04A = 0\Big) \qquad \text{or} \qquad \Big(e^{0.04t}A\Big)' = 0.$$

After integrating, we get $e^{0.04t}A = K$, or $A(t) = Ke^{-0.04t}$, as before. The initial condition ($A = 20$ when $t = 0$) gives us $A(t) = 20e^{-0.04t}$.

(b) How long will it take before there is only 5 pounds of salt left in the tank?

Solution: This question asks us to find t so that $A(t) = 5$, or $20e^{-0.04t} = 5$. That is, we must find t so that $e^{-0.04t} = \frac{5}{20} = 0.25$. Taking natural logs, this is equivalent to $-0.04t = \ln(0.25)$, or $t = -25\ln(0.25) \approx 34.6574$ minutes.

5 Is $y = e^{2x}$ a solution of the differential equation $y'' - 3y' + 2y = 5e^{2x}$? Fully justify your answer.

Solution: The function $y = e^{2x}$ is *not* a solution of the differential equation $y'' - 3y' + 2y = 5e^{2x}$. We verify this by computing some derivatives:

$$y = e^{2x} \qquad y' = 2e^{2x} \qquad y'' = 4e^{2x}.$$

Plugging these into the equation gives us

$$\begin{aligned} y'' - 3y' + 2y &= \left(4e^{2x}\right) - 3\left(2e^{2x}\right) + 2\left(e^{2x}\right) \\ &= 4e^{2x} - 6e^{2x} + 2e^{2x} \\ &= 0e^{2x} = 0. \end{aligned}$$

In particular, $y'' - 3y' + 2y \neq 5e^{2x}$ for this particular function $y = e^{2x}$.