1 Alice is waiting for a Circle Link shuttle bus (a "greenie"). Her experience is that she waits an average of 10 minutes for each bus. She is a Math 126 student, so she models her waiting time as an exponential random variable.

(a) What is the probability (under this model) that Alice waits 5 minutes or less for a bus?

Solution: Recall that the exponential density function is $f(x) = \lambda e^{-\lambda x}$ (defined for $x \geq 0$). We find the parameter λ by using the fact that the expected value is $E(X) = 1/\lambda$. Since the expected value here is 10 minutes, we get $1/\lambda = 10$ or $\lambda = \frac{1}{10} = 0.1$.

The probability that X (the random variable with this exponential distribution) is less than 5 is

$$\Pr(X \leq 5) = \int_0^5 \lambda e^{-\lambda x}\, dx = \int_0^5 0.1e^{-0.1x}\, dx.$$

Using the integration rule $\int e^{rx}\, dx = \frac{1}{r}e^{rx} + K$, we get

$$\Pr(X \leq 5) = \int_0^5 0.1e^{-0.1x}\, dx = 0.1 \cdot \frac{1}{-0.1}e^{-0.1x}\Big|_0^5 = -\left(e^{-0.1(5)} - e^{-0.1(0)}\right) = -\left(e^{-0.5} - 1\right).$$

Thus the probability that Alice waits 5 minutes or less for a bus is $\Pr(X \leq 5) = 1 - e^{-0.5} \approx 0.3935$ or about 39.35%.

(b) Suppose Alice has an exam this morning. She'll make it to the exam on time if the bus arrives within 20 minutes. What is the probability that Alice is late (that is, **not** on time) to her exam?

Solution: This question asks for $\Pr(X \geq 20)$, where X is the same exponentially distributed random variable as in part (a). We can compute this probability as

$$\Pr(X \geq 20) = \int_{20}^{\infty} \lambda e^{-\lambda x}\, dx = \int_{20}^{\infty} 0.1e^{-0.1x}\, dx.$$

This is slightly complicated by the fact that this is an improper integral (there's an infinity as an upper limit). Rather than deal with the limit involved in this computation, it is simpler to deal with the complementary probability:

$$\Pr(X \geq 20) = 1 - \Pr(X \leq 20) = 1 - \int_0^{20} \lambda e^{-\lambda x}\, dx = 1 - \int_0^{20} 0.1e^{-0.1x}\, dx.$$

This last integral is straightforward to calculate (and very similar to the integral in part (a)):

$$\int_0^{20} 0.1e^{-0.1x}\, dx = 0.1 \cdot \frac{1}{-0.1}e^{-0.1x}\Big|_0^{20} = -\left(e^{-0.1(20)} - e^{-0.1(0)}\right) = -\left(e^{-2} - 1\right) = -e^{-2} + 1.$$

Thus

$$\Pr(X \geq 20) = 1 - \int_0^{20} 0.1e^{-0.1x}\, dx = 1 - \left(-e^{-2} + 1\right) = e^{-2}.$$

That is, the probability that Alice is late to her exam is $\Pr(X \geq 20) = e^{-2} \approx 0.1353$ or about 13.53%.

2 The SACT is a new test for college-bound high school students. It is designed so that scores are normally distributed with a mean of 400 and a standard deviation of 50.

(a) Find the probability that a randomly selected high school student scores between 300 and 450 on the SACT.

Solution: Let X be the normal random variable with mean $\mu = 400$ and standard deviation $\sigma = 50$. This question asks for $\Pr(300 \leq X \leq 450)$. We translate this into a question about the standard normal random variable in the usual way:

$$\begin{aligned}\Pr(300 \leq X \leq 450) &= \Pr\left(\frac{300-\mu}{\sigma} \leq Z \leq \frac{450-\mu}{\sigma}\right)\\ &= \Pr\left(\frac{300-400}{50} \leq Z \leq \frac{450-400}{50}\right)\\ &= \Pr\left(-2 \leq Z \leq 1\right).\end{aligned}$$

Now we use the symmetry of the normal distribution to split this into two probabilities on our table:

$$\begin{aligned}\Pr(300 \leq X \leq 450) &= \Pr\left(-2 \leq Z \leq 1\right)\\ &= \Pr\left(-2 \leq Z \leq 0\right) + \Pr\left(0 \leq Z \leq 1\right)\\ &= \Pr\left(0 \leq Z \leq 2\right) + \Pr\left(0 \leq Z \leq 1\right) \qquad \text{by symmetry.}\end{aligned}$$

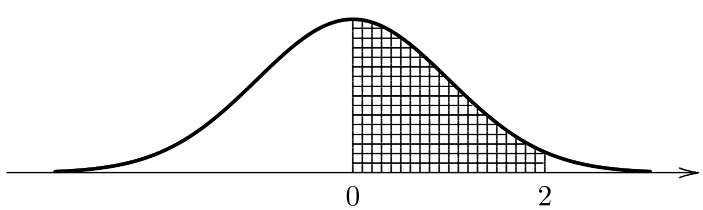
From the standard normal distribution table, we see that $\Pr(0 \leq Z \leq 2) = 0.4772$ and that $\Pr(0 \leq Z \leq 1) = 0.3413$. Thus $\Pr(300 \leq X \leq 450) = 0.4772 + 0.3413 = 0.8185$, so the probability that a randomly selected high school student scores between 300 and 450 on the SACT is about 81.85%.

(b) Find the probability that a randomly selected high school student scores over 500 on the SACT.

Solution: This asks for $\Pr(X \geq 500)$, where X is a normal random variable with mean $\mu = 400$ and standard deviation $\sigma = 50$. We change this question to a question about the standard normal distribution in the usual way:

$$\Pr(X \geq 500) = \Pr\left(Z \geq \frac{500-\mu}{\sigma}\right) = \Pr\left(Z \geq \frac{500-400}{50}\right) = \Pr\left(Z \geq 2\right).$$

We can read the probability $\Pr(0 \leq Z \leq 2) = 0.4772$ from the standard normal distribution table. Here's a picture of this probability thought of the area under the standard normal density curve:



We're interested in the area under the curve to the right of 2. Since the area to the right of 0 is 0.5, the area past 2 is $0.5 - 0.4772 = 0.0228$. Written in terms of probabilities, this says

$$\Pr(Z \geq 2) = \Pr(Z \geq 0) - \Pr(0 \leq Z \leq 2) = 0.5 - 0.4772 = 0.0228.$$

Thus the probability that a randomly selected high school student scores over 500 on the SACT is about 2.28%.

(c) A college wants to be selective and so only decides to accept only students who score in the top 6% on the SACT. What is the lowest score on the test that this college should consider? (Please round your answer to the nearest multiple of 10.)

Solution: This question asks us to find x so that $\Pr(X \geq x) = 0.06$, where X is again a normal distribution with mean $\mu = 400$ and standard deviation $\sigma = 50$. Again we can turn this into a question about the standard normal distribution:

$$0.06 = \Pr(X \geq x) = \Pr\left(Z \geq \frac{x-\mu}{\sigma}\right) = \Pr\left(Z \geq \frac{x-400}{50}\right).$$

Thus we're looking for $z = \frac{x-400}{50}$ so that $\Pr(Z \geq z) = 0.06$. As in part (b), this means we're looking for z with $\Pr(0 \leq Z \leq z) = 0.44$:

$$\Pr(0 \leq Z \leq z) = \Pr(Z \geq 0) - \Pr(Z \geq z) = 0.5 - 0.06 = 0.44.$$

We turn to our trusty standard normal distribution table to find a Z-value of precisely 0.4400. Unfortunately, there is no such z on the table:

z	1.55	z	1.56
Z-value	0.4394	0.4400	0.4406

We can use a nearby value or, do what is often done in this situation – interpolate between 1.55 and 1.56 to find the z with Z-value 0.4400:

$$\frac{\Delta z}{\Delta \text{ Value}} = \frac{z - 1.55}{0.4400 - 0.4394} = \frac{1.56 - 1.55}{0.4406 - 0.4394} \quad \text{or} \quad \frac{z-1.55}{0.0006} = \frac{0.01}{0.0012}.$$

Thus $z = 1.55 + \frac{0.01}{0.0012}(0.0006) = 1.555$.
Solving for x in $z = \frac{x-\mu}{\sigma} = \frac{x-400}{50}$, we get the following possible values:

z	x
1.55	477.5
1.555	477.75
1.56	478

Since we're asked to round to nearest multiple of 10, these different z's all give us the same answer: the lowest score in the top 6% is about 480.

3 Find the value, if any, of each of the following improper integrals. Justify your answers using limits!

(a) $\int_0^{\infty} 12e^{-4x}\,dx$

Solution: This is integral is defined as

$$\int_0^{\infty} 12e^{-4x}\,dx = \lim_{b\to\infty} \int_0^b 12e^{-4x}\,dx.$$

We can compute the integral using the fact that $\int e^{rx}\,dx = \frac{1}{r}e^{rx} + K$:

$$\begin{aligned}\int_0^{\infty} 12e^{-4x}\,dx &= \lim_{b\to\infty} 12 \cdot \frac{1}{-4}e^{-4x}\Big|_0^b \\ &= \lim_{b\to\infty} -3\left(e^{-4b} - e^{-4(0)}\right) \\ &= -3\,(0-1) = 3.\end{aligned}$$

Thus the value of the improper integral is 3.

(b) $\int_1^5 \frac{1}{(x-1)^2}\,dx$

Solution: This is an improper integral because the function $\frac{1}{(x-1)^2}$ is undefined at $x = 1$. Thus, by definition, the the improper integral is a limit:

$$\int_1^5 \frac{1}{(x-1)^2}\,dx = lim_{a\to 1^+} \int_a^5 \frac{1}{(x-1)^2}\,dx.$$

The integral is not hard to compute. We use the substitution $u = x - 1$, so $du = dx$ and

$$\int \frac{1}{(x-1)^2}\,dx = \int \frac{1}{u^2}\,du = \int u^{-2}\,du = -u^{-1} + K = -\frac{1}{u} + K = -\frac{1}{x-1} + K.$$

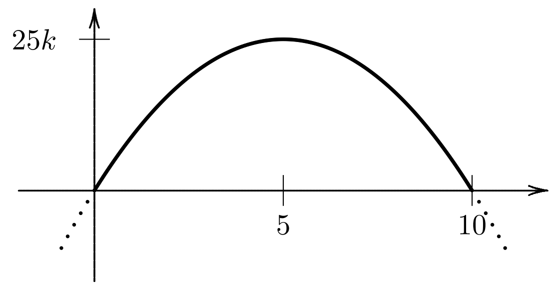
Thus the improper integral is

$$\begin{aligned}\int_1^5 \frac{1}{(x-1)^2}\,dx &= \lim_{a\to 1^+} \int_a^5 \frac{1}{(x-1)^2}\,dx \\ &= \lim_{a\to 1^+} \left(-\frac{1}{x-1}\right)\Big|_a^5 \\ &= \lim_{a\to 1^+} \left[\left(-\frac{1}{5-1}\right) - \left(-\frac{1}{a-1}\right)\right] \\ &= \lim_{a\to 1^+} \left(-\frac{1}{4} + \frac{1}{a-1}\right).\end{aligned}$$

Thus the improper integral does not converge since this limit is unbounded: as a decreases to 1, $a - 1$ decreases to zero, so $\frac{1}{a-1}$ grows without bound. That is, the given improper integral is divergent.

4 (a) Find k, if possible, so that $f(x) = k(10x - x^2)$ is a probability density function on the interval $0 \leq x \leq 10$. If it is not possible, explain why not.

Solution: Recall that for $f(x)$ to be a probability density function on the interval $0 \leq x \leq 10$, it must satisfy two conditions. First, we must have $f(x) \geq 0$ on this interval, and the total probability must be 1: $\int_0^{10} f(x)\,dx = 1$. Here's a graph of the curve showing that $f(x) \geq 0$ on the interval:



The constant k is really determined by the integral. We compute:

$$\begin{aligned} 1 &= \int_0^{10} k(10x - x^2)\,dx \\ &= k\left(5x^2 - \frac{1}{3}x^3\right)\Big|_0^{10} \\ &= k\left[\left(5(10)^2 - \frac{1}{3}(10)^3\right) - \left(5(0)^2 - \frac{1}{3}(0)^3\right)\right] \\ &= k\left(500 - \frac{1000}{3}\right) = \frac{500}{3}k. \end{aligned}$$

Thus $k = \frac{3}{500} = 0.006$.

(b) Suppose $\{-8, 2, 6, x, y\}$ is a set of population data with $\mu = 1$ and $\sigma^2 = 22.4$. Find x and y.

Solution: That the mean is 1 means that

$$\frac{-8 + 2 + 6 + x + y}{5} = 1 \qquad \text{or} \qquad x + y = 5.$$

That the variance is $\sigma^2 = 22.4$ means that

$$\frac{(-8-1)^2 + (2-1)^2 + (6-1)^2 + (x-1)^2 + (y-1)^2}{5} = 22.4$$

or

$$81 + 1 + 25 + (x-1)^2 + (y-1)^2 = 5 \cdot 22.4 = 112.$$

This simplifies to $(x-1)^2 + (y-1)^2 = 5$. We make the substitution $y = 5 - x$ (from the mean calculation) to get

$$(x-1)^2 + (5 - x - 1)^2 = 5 \qquad \text{or} \qquad x^2 - 2x + 1 + 16 - 8x + x^2 = 5.$$

This quadratic simplifies to $2x^2 - 10x + 12 = 0$, or $x^2 - 5x + 6 = 0$. This factors into $(x-3)(x-2) = 0$, so it has roots $x = 2$ and $x = 3$. These two values of x are our x and y.

5 Suppose X is a random variable with probability density function $f(x) = \dfrac{9}{4x^3}$ on the interval $1 \le x \le 3$.

(a) Calculate the probability $\Pr(X \le 2)$.

Solution: By definition, this probability is

$$\Pr(X \le 2) = \int_1^2 \frac{9}{4x^3}\,dx.$$

We calculate this integral:

$$\begin{aligned}\Pr(X \le 2) &= \frac{9}{4}\int_1^2 x^{-3}\,dx = \frac{9}{4}\cdot\frac{1}{-2}x^{-2}\Big|_1^2\\ &= -\frac{9}{8}\left(2^{-2} - 1^{-2}\right) = -\frac{9}{8}\left(\frac{1}{4} - 1\right)\\ &= -\frac{9}{8}\left(-\frac{3}{4}\right) = \frac{27}{32} = 0.84375.\end{aligned}$$

Thus this probability is $\Pr(X \le 2) = 0.84375$ or 84.375%.

(b) Calculate the expected value $E(X)$.

Solution: By definition, the expected value is

$$E(X) = \int_1^3 xf(x)\,dx = \int_1^3 x\cdot\frac{9}{4x^3}\,dx = \frac{9}{4}\int_1^3 x^{-2}\,dx.$$

We compute this integral:

$$\begin{aligned}E(X) &= \frac{9}{4}\int_1^3 x^{-2}\,dx = \frac{9}{4}\left(\frac{1}{-1}x^{-1}\right)\Big|_1^3\\ &= \frac{9}{4}\left[\left(-3^{-1}\right) - \left(-1^{-1}\right)\right] = \frac{9}{4}\left(-\frac{1}{3} + 1\right)\\ &= \frac{9}{4}\cdot\frac{2}{3} = \frac{3}{2} = 1.5.\end{aligned}$$

Thus the expected value is 1.5.

(c) Calculate the variance $\mathrm{Var}(X)$.

Solution: By definition, the variance is

$$\mathrm{Var}(X) = E(X^2) - E(X)^2 = \int_1^3 x^2 f(x)\,dx - \mu^2 = \int_1^3 x^2\cdot\frac{9}{4x^3}\,dx - 1.5^2 = \frac{9}{4}\int_1^3 x^{-1}\,dx - 2.25.$$

We compute this integral:

$$\begin{aligned}E(X^2) &= \frac{9}{4}\int_1^3 x^{-1}\,dx = \frac{9}{4}\ln(x)\Big|_1^3\\ &= \frac{9}{4}\left(\ln(3) - \ln(1)\right) = \frac{9}{4}\ln(3).\end{aligned}$$

Thus the variance is $\mathrm{Var}(X) = \frac{9}{4}\ln(3) - 2.25 \approx 0.2219$.