1 Evaluate the following indefinite integrals:

(a) $\int x\sin(5x)\ dx$

Solution: This is an integral that requires integration by parts. We'll choose $u = x$ and $dv = \sin(5x)\ dx$, so $u\ dv = x\sin(5x)\ dx$ is what we're integrating. Then $du = dx$ and $v = -\frac{1}{5}\cos(5x)$. That last integral needs a little explanation. We make the substitution $t = 5x$, so $dt = 5\ dx$ or $dx = \frac{1}{5}\ dt$. Then

$$v = \int dv = \int \sin(5x)\ dx = \int \sin(t)\cdot\frac{1}{5}\ dt = \frac{1}{5}\int \sin(t)\ dt = -\frac{1}{5}\cos(t) + K = -\frac{1}{5}\cos(5x) + K.$$

Of course, for the integration by parts we choose the simplest v possible (with $K = 0$).
Thus

$$\begin{aligned}\int x\sin(5x)\ dx &= uv - \int v\ du = x\left(-\frac{1}{5}\cos(5x)\right) - \int\left(-\frac{1}{5}\cos(5x)\right)\ dx\\ &= -\frac{1}{5}x\cos(5x) + \frac{1}{5}\int\cos(5x)\ dx\\ &= -\frac{1}{5}x\cos(5x) + \frac{1}{25}\sin(5x) + K.\end{aligned}$$

(This last integral uses the same substitution $t = 5x$ as in our computation of v, above.) This is our answer.

(b) $\int xe^{3x^2}\ dx$

Solution: This integral requires only the substitution $u = 3x^2$. From this we get $du = 6x\ dx$, or $dx = \frac{1}{6x}\ du$. Thus

$$\int xe^{3x^2}\ dx = \int xe^u\cdot\frac{1}{6x}\ du = \frac{1}{6}\int e^u\ du.$$

This function is easy to integrate: we get $\frac{1}{6}e^u + K$ or, in terms of x, $\frac{1}{6}e^{3x^2} + K$. This is our final answer.

(c) $\int 3x^2e^x\ dx$

Solution: To compute this integral, we must integrate by parts twice. We start with the choice $u = 3x^2$ and $dv = e^x\ dx$ (so $u\ dv = 3x^2e^x\ dx$). From this $du = 6x\ dx$ and $v = e^x$, so

$$\int 3x^2e^x\ dx = 3x^2e^x - \int e^x\cdot 6x\ dx = 3x^2e^x - \int 6xe^x\ dx.$$

Thus we've improved our situation: our original integral involved x^2e^x, but the new integral involves only xe^x. We now use integration by parts again, this time with $u = 6x$ and $dv = e^x\ dx$. Thus $du = 6\ dx$ and $v = e^x$, so

$$\begin{aligned}\int 3x^2e^x\ dx &= 3x^2e^x - \int 6xe^x\ dx\\ &= 3x^2e^x - \left(6xe^x - \int e^x\cdot 6\ dx\right)\\ &= 3x^2e^x - 6xe^x + 6\int e^x\ dx.\end{aligned}$$

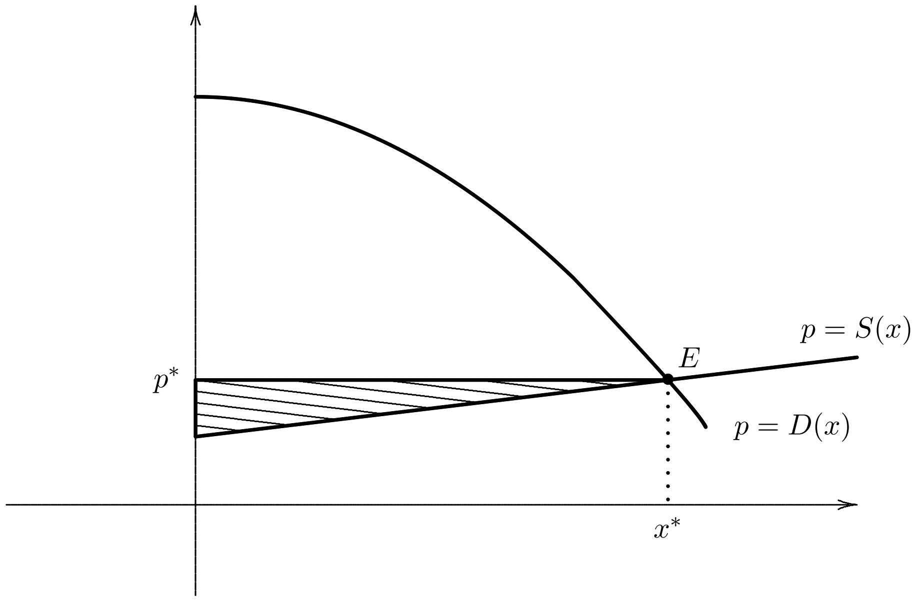
Integrating this last integral, we get

$$\int 3x^2e^x\ dx = 3x^2e^x - 6xe^x + 6e^x + K.$$

This is our final answer.

2 (a) The following picture shows a demand curve $p = D(x)$ and a supply curve $p = S(x)$. Shade in the area on this graph that represents the Producer's Surplus (PS).

Solution: In the graph below, I have added the equilibrium point $E = (x^*, p^*)$ and shaded the region corresponding to the Producer's Surplus (PS):



(b) If $p = D(x) = 18 - 2x^2$ and $p = S(x) = x + 3$, calculate the Consumer's Surplus (CS). (Note that these are the functions graphed in part (a), but this is **not** what you shaded in part (a).)

Solution: The Consumer's Surplus (CS) is, by the formula given on the formula sheet,

$$CS = \int_0^{x^*} D(x)\, dx - x^* p^*,$$

where $E = (x^*, p^*)$ is the equilibrium point (the intersection of the supply and demand curves). We begin by finding the intersection point. This is where $D(x) = S(x)$, or $18 - 2x^2 = x + 3$. If we write this as $0 = 2x^2 + x - 15$, we can either factor into $0 = (2x - 5)(x + 3)$ or use the quadratic formula to find the roots: $x = 5/2$ or $x = -3$. Since the equilibrium point must be at a positive quantity x, we get $x^* = 5/2 = 2.5$. At this point, $p^* = D(x^*) = S(x^*) = 5.5$.

Now our formula gives

$$\begin{aligned} CS &= \int_0^{x^*} D(x)\, dx - x^* p^* \\ &= \int_0^{2.5} \left(18 - 2x^2\right)\, dx - (2.5)(5.5) \\ &= \left(18x - \frac{2}{3}x^3\right)\Big|_0^{2.5} - 13.75 \\ &= \left(18(2.5) - \frac{2}{3}(2.5)^3\right) - (0 - 0) - 13.75 \\ &= \frac{125}{6} \approx 20.8333. \end{aligned}$$

This is our Consumer's Surplus.

3 The depreciation rate of a new car obeys the equation

$$V'(t) = \frac{-12,000}{(3t+2)^2}, \qquad 0 \le t \le 5$$

where V is the value of the car t years after it was purchased. If the car cost \$18,000 new, what is its value after 3 years?

Solution: This question asks for $V(3)$, the value of the car after 3 years. We begin by integrating $V'(t)$ to find $V(t)$:

$$V(t) = \int V'(t)\, dt = \int \frac{-12,000}{(3t+2)^2}\, dt.$$

This is easiest to calculate if we make the substitution $u = 3t+2$, so $du = 3\, dt$ or $dt = \frac{1}{3}\, du$. Thus

$$\int \frac{-12,000}{(3t+2)^2}\, dt = -12,000 \int \frac{1}{u^2} \cdot \frac{1}{3}\, du = \frac{-12,000}{3} \int \frac{1}{u^2}\, du = -4,000 \int u^{-2}\, du$$
$$= -4,000 \cdot \frac{u^{-1}}{-1} + K = 4,000 \cdot \frac{1}{u} + K = \frac{4,000}{3t+2} + K.$$

Thus $V(t) = \frac{4,000}{3t+2} + K$.

To find K, we plug in the initial condition: $V(0) = \$18,000$ (that is, $V = 18,000$ when $t = 0$). We get $18,000 = \frac{4,000}{3(0)+2} + K = 2,000 + K$, or $K = 16,000$. Hence $V(t) = \frac{4,000}{3t+2} + 16,000$.

We now simply plug in $t = 3$ years to find the value

$$V(3) = \frac{4,000}{3(3)+2} + 16,000 \approx \$16,363.64.$$

Thus the car is worth about \$16,363.64 after 3 years.

4 Suppose the population of a town is projected to be $P(t) = 7000e^{0.04t}$ people, where t is measured in years from today. Find the average population in this city over the next ten years. If you round, then please round your answer to the nearest person.

Solution: This question asks for the average population from $t = 0$ to $t = 10$ years. Since the population is given as the function $P(t) = 7000e^{0.04t}$ people, we simply want to average this function over the given time interval. By the formula for average value, we have

$$\text{Average Population} = \frac{1}{10-0}\int_0^{10} 7000e^{0.04t}\,dt = \frac{7000}{10}\int_0^{10} e^{0.04t}\,dt = 700\int_0^{10} e^{0.04t}\,dt.$$

This integral is fairly straightforward: either we can make the substitution $u = 0.04t$ or use the formula given to us on the formula sheet:

$$\int e^{ax}\,dx = \frac{1}{a}e^{ax} + K.$$

In either case, we get

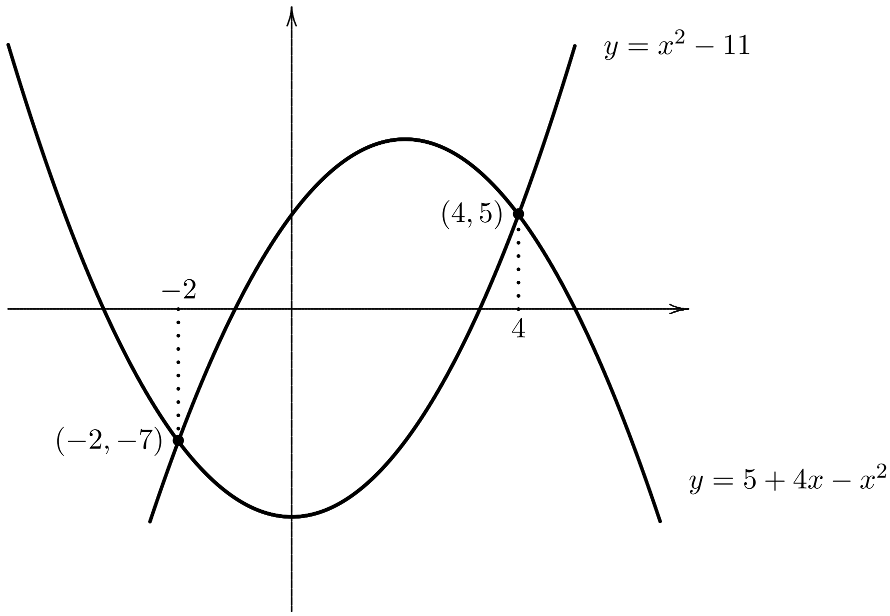
$$\begin{aligned}\text{Average Population} &= 700\int_0^{10} e^{0.04t}\,dt = 700 \cdot \frac{1}{0.04}e^{0.04t}\Big|_0^{10} \\ &= \frac{700}{0.04}\left(e^{0.04(10)} - e^{0.04(0)}\right) = 17,500\left(e^{0.4} - 1\right) \approx 8,606.93.\end{aligned}$$

Thus the average population is roughly $8,607$ people.

5 Find the area between the graphs of the functions

$$y = 5 + 4x - x^2 \qquad \text{and} \qquad y = x^2 - 11.$$

For your convenience, the graphs of these functions are shown below:



Solution: In order to calculate the area bounded by these two curves, we need to know where they intersect. We find this by setting the two equations equal to each other:

$$5 + 4x - x^2 = x^2 - 11.$$

We move everything to one side to get $0 = 2x^2 - 4x - 16$. This factors into $0 = 2(x-4)(x+2)$, so the roots are $x = 4$ and $x = -2$. These values correspond to the points $(x, y) = (4, 5)$ and $(x, y) = (-2, -7)$, as noted on the graph.

The area between the curves is then given by

$$\text{Area} = \int_{-2}^{4} (\text{ top curve } - \text{ bottom curve }) \, dx$$

or (since $y = 5 + 4x - x^2$ is the top curve between $x = -2$ and $x = 4$)

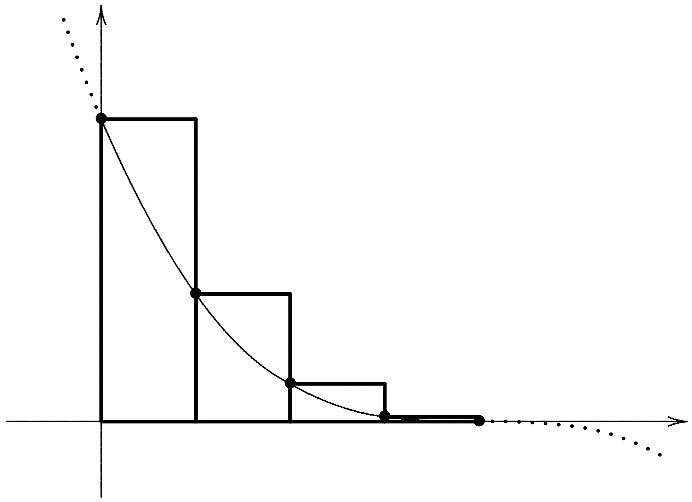
$$\text{Area} = \int_{-2}^{4} \left((5 + 4x - x^2) - (x^2 - 11)\right) \, dx = \int_{-2}^{4} \left(16 + 4x - 2x^2\right) \, dx.$$

This integral is straightforward:

$$\begin{aligned} \text{Area} &= \int_{-2}^{4} \left(16 + 4x - 2x^2\right) \, dx = \left(16x + 2x^2 - \frac{2}{3}x^3\right) \Big|_{-2}^{4} \\ &= \left(16(4) + 2(4)^2 - \frac{2}{3}(4)^3\right) - \left(16(-2) + 2(-2)^2 - \frac{2}{3}(-2)^3\right) \\ &= \left(64 + 32 - \frac{128}{3}\right) - \left(-32 + 8 + \frac{16}{3}\right) \\ &= 72. \end{aligned}$$

Thus the area we want is 72 square units.

6 Below is the graph of a curve $y = f(x)$. We would like to estimate the area under this curve in the first quadrant (from $x = 0$ to $x = 4$). Unfortunately, we do not know the equation of this curve, but only the values shown in the table below (and as dots on the graph):



x	$f(x)$
0	16
1	7
2	2
3	$1/2$
4	0

(a) Approximate the area under $y = f(x)$ from $x = 0$ to $x = 4$ by partitioning $[0, 4]$ into four subintervals of equal length and choosing u as the left endpoint of each subinterval.

Solution: I have drawn four rectangles on the curve, above. These rectangles contain the area asked for – when we say "choosing u as the left endpoint of each subinterval" we mean that the height of each rectangle is determined by the height of the curve over the left endpoint of the subinterval. This is the way I've drawn the rectangles.

To estimate the area under the curve, we calculate the area of these rectangles. This is fairly simple: they all have width $\Delta x = \frac{b-a}{n} = \frac{4-0}{4} = 1$. Thus the total area of these rectangles is

$$f(0) \cdot \Delta x + f(1) \cdot \Delta x + f(2) \cdot \Delta x + f(3) \cdot \Delta x = 16 \cdot 1 + 7 \cdot 1 + 2 \cdot 1 + \frac{1}{2} \cdot 1 = 25.5.$$

That is, the area under the curve $y = f(x)$ from $x = 0$ to $x = 4$ is about 25.5 square units.

(b) Is your estimate in part (a) is too high or too low? Explain your reasoning.

Solution: The estimate in part (a) is too high. This is clear from the picture: the rectangles cover *more* than simply the area underneath the curve. This happens because the function is decreasing on the interval $[0, 4]$, so the left endpoint of each subinterval is the highest the curve gets on that subinterval.