1 Find the following indefinite integrals. Show all work necessary for the method you are using.

(a) $\displaystyle\int \left(\frac{5}{x} + 10\cos(2x)\right)\, dx$

Solution: By the properties of integrals, this integral can be written as the some of two integrals:

$$\int \left(\frac{5}{x} + 10\cos(2x)\right)\, dx = 5\int \frac{1}{x}\, dx + 10\int \cos(2x)\, dx.$$

The first integral $\int \frac{1}{x}\, dx$ is one of our basic integrals (it integrates to $\ln(|x|)$ plus a constant), but the second one $\int \cos(2x)\, dx$ isn't. For it, we use the substitution $u = 2x$, so $du = 2\, dx$ or $dx = \frac{1}{2}\, du$. Thus

$$\begin{aligned}\int \left(\frac{5}{x} + 10\cos(2x)\right)\, dx &= 5\int \frac{1}{x}\, dx + 10\int \cos(u)\cdot\frac{1}{2}\, du \\ &= 5\int \frac{1}{x}\, dx + 5\int \cos(u)\, du \\ &= 5\ln(|x|) + 5\sin(u) + K \\ &= 5\ln(|x|) + 5\sin(2x) + K.\end{aligned}$$

This is our final answer.

(b) $\displaystyle\int 8xe^{2x}\, dx$

Solution: For this integral, we again make the substitution $t = 2x$, in the hopes of turning the integral into a multiple of $\int e^t\, dt$. Thus $dt = 2\, dx$, or $dx = \frac{1}{2}\, dt$. This turns our integral into

$$\int 8xe^{2x}\, dx = \int 8\cdot\frac{1}{2}te^t\cdot\frac{1}{2}\, dt = 2\int te^t\, dt.$$

Now it becomes clear why we used t for our substitution rather than u: this integral requires integration by parts. Let $u = t$ and $dv = e^t\, dt$, so

$$\begin{aligned} u &= t & dv &= e^t\, dt \\ du &= dt & v &= e^t. \end{aligned}$$

Thus

$$\begin{aligned}\int 8xe^{2x}\, dx &= 2\int te^t\, dt \\ &= 2\left(te^t - \int e^t\, du\right) \\ &= 2\left(te^t - e^t\right) + K \\ &= 2te^t - 2e^t + K.\end{aligned}$$

Finally, we replace t with $2x$ (our original substitution) to get our final answer:

$$\int 8xe^{2x}\, dx = 4xe^{2x} - 2e^{2x} + K.$$

(c) $\int 8xe^{x^2}\,dx$

Solution: For this problem we make the substitution $u = x^2$, hoping to end up with an integral that is a multiple of $\int e^u\,du$. Thus $du = 2x\,dx$, or $dx = \frac{1}{2x}\,du$. Hence

$$\int 8xe^{x^2}\,dx = \int 8xe^u \cdot \frac{1}{2x}\,du = 4\int e^u\,du.$$

This integrates to $4e^u + K$, so our final answer (after substituting back in $u = x^2$) is

$$\int 8xe^{x^2}\,dx = 4e^{x^2} + K.$$

2 The value of a certain computer system is changing at the rate of

$$V'(t) = -3200\left(7 + 6t - t^2\right) \qquad 0 \le t \le 7.$$

(a) How much does the value change during the first three years?

Solution: The change in value during the first three years is $V(3) - V(0)$. This is also the value of the definite integral

$$\text{change in value} = \int_0^3 V'(t)\, dt = -3200 \int_0^3 \left(7 + 6t - t^2\right)\, dt.$$

We integrate to get

$$\begin{aligned}\text{change in value} &= -3200\left[7t + 3t^2 - \frac{1}{3}t^3\right]_0^3 \\ &= -3200\left[\left(7(3) + 3(3)^2 - \frac{1}{3}(3)^3\right) - \left(7(0) + 3(0)^2 - \frac{1}{3}(0)^3\right)\right] \\ &= -124,800.\end{aligned}$$

Thus the computer system has lost \$124,800 in value during the first three years.

(b) If the computer system was originally worth \$1,200,000, how much is it worth after the first five years?

Solution: For this question, it is perhaps simplest to find the function $V(t)$ before trying to find the value $V(5)$. To find $V(t)$, we simply anti-differentiate (that is, integrate) $V'(t)$. Of course, we've already done this in part (a):

$$V(t) = \int V'(t)\, dt = -3200 \int \left(7 + 6t - t^2\right)\, dt = -3200\left(7t + 3t^2 - \frac{1}{3}t^3\right) + K.$$

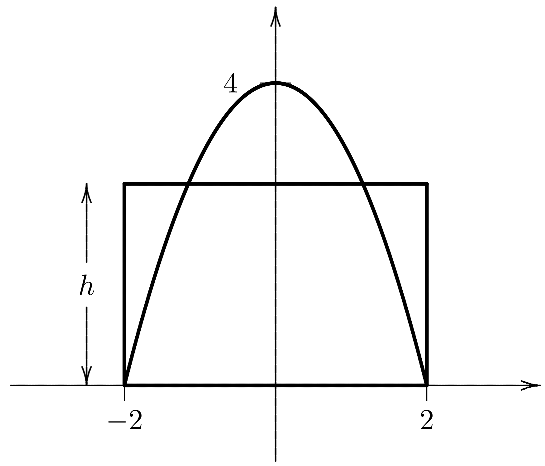
We can plug in $V(0) = \$1,200,000$ to find K:

$$\$1,200,000 = -3200\left(7(0) + 3(0)^2 - \frac{1}{3}(0)^3\right) + K \qquad \Longrightarrow \qquad K = \$1,200,000.$$

Thus $V(t) = 1,200,000 - 3200\left(7t + 3t^2 - \frac{1}{3}t^3\right)$, and so the value after the first five years is

$$V(5) = 1,200,000 - 3200\left(7(5) + 3(5)^2 - \frac{1}{3}(5)^3\right) \approx \$981,333.33.$$

3 (a) Suppose the area bounded by the curve $f(x) = 4 - x^2$ and the x-axis is the same as the area of the rectangle shown below:



Find the height h of the rectangle.

Solution: The height h is the average value of $f(x)$ from $x = -2$ to $x = 2$. One way to see this is simply to compute both areas directly: the area bounded by the curve $f(x) = 4 - x^2$ and the x-axis is simply

$$\int_{-2}^{2} (4 - x^2)\, dx.$$

On the other hand, the area of the rectangle is its height h times its width $2 - (-2) = 4$. Thus this area is $4h$, and so

$$4h = \int_{-2}^{2} (4 - x^2)\, dx \qquad \text{or} \qquad h = \frac{1}{4}\int_{-2}^{2} (4 - x^2)\, dx.$$

(This is precisely the average value of $f(x)$, as claimed.)

We now compute h from the integral, above:

$$\begin{aligned} h &= \frac{1}{4}\int_{-2}^{2} (4 - x^2)\, dx \\ &= \frac{1}{4}\left[4x - \frac{1}{3}x^3\right]_{-2}^{2} \\ &= \frac{1}{4}\left[\left(4(2) - \frac{1}{3}(2)^3\right) - \left(4(-2) - \frac{1}{3}(-2)^3\right)\right] \\ &= \frac{8}{3} \approx 2.6667. \end{aligned}$$

(b) Suppose $\int_{1}^{3} g(x)\, dx = 10$. Without knowing anything else about $g(x)$, determine the average value of $g(x)$ on the interval $[1, 3]$.

Solution: By the formula given on the formula sheet, the average value of $g(x)$ on the interval $[1, 3]$ is

$$\text{A.V.} = \frac{1}{3-1}\int_{1}^{3} g(x)\, dx = \frac{1}{2} \cdot 10 = 5.$$

Thus the average value of $g(x)$ on the interval $[1, 3]$ is 5.

4 The marginal cost function for producing x units of a product is

$$C'(x) = 200 - 100x^{-0.1}.$$

Find the increase in total cost if production increases from 100 to 150 units.

Solution: The increase in total cost is $C(150) - C(100)$. This is, of course, the definite integral of $C'(x)$ from $x = 100$ to $x = 150$:

$$\text{increase in cost} = \int_{100}^{150} C'(x)\, dx = \int_{100}^{150} \left(200 - 100x^{-0.1}\right)\, dx.$$

We can integrate the corresponding indefinite integral:

$$\begin{aligned}\int \left(200 - 100x^{-0.1}\right)\, dx &= 200 \int dx - 100 \int x^{-0.1}\, dx \\ &= 200x - 100 \cdot \frac{x^{0.9}}{0.9} + K \\ &= 200x - \frac{1000}{9} x^{0.9} + K.\end{aligned}$$

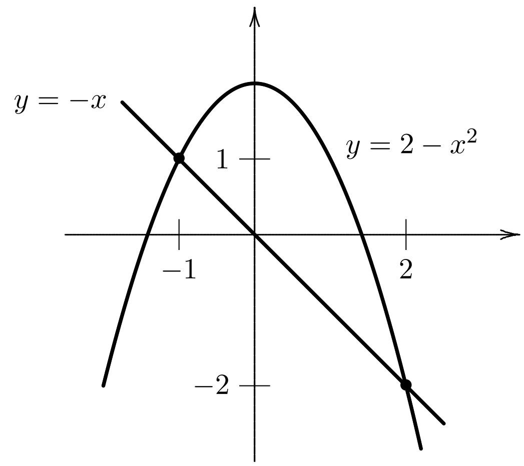
For the definite integral, we can choose any anti-derivative. In particular, we'll choose the simplest (the one with $K = 0$). Thus

$$\begin{aligned}\text{increase in cost} &= \int_{100}^{150} \left(200 - 100x^{-0.1}\right)\, dx \\ &= \left[200x - \frac{1000}{9} x^{0.9}\right]_{100}^{150} \\ &= \left[\left(200(150) - \frac{1000}{9}(150)^{0.9}\right) - \left(200(100) - \frac{1000}{9}(100)^{0.9}\right)\right] \\ &= 10000 - \frac{1000}{9}\left((150)^{0.9} - (100)^{0.9}\right) \\ &\approx 6,912.54.\end{aligned}$$

Thus the costs have increased about \$6,912.54.

5 (a) Find the area bounded by the parabola $y = 2 - x^2$ and the straight line $y = -x$.

Solution: It always helps in these problems to draw a picture:

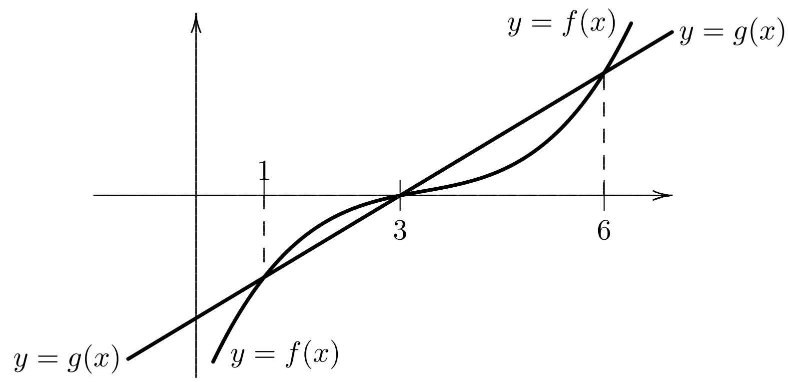


From the picture we can see the the parabola $y = 2 - x^2$ and the line $y = -x$ bound a region from $x = -1$ to $x = +2$. (We can also confirm this algebraically: the curves cross when $2 - x^2 = -x$, or $0 = x^2 - x - 2$, or $(x-2)(x+1) = 0$. Thus they cross when $x = -1$ or $x = +2$.) The area bounded by the two curves is therefore

$$\begin{aligned}
\text{Area bounded by the curves} &= \int_{-1}^{2} (\text{top curve} - \text{bottom curve})\ dx \\
&= \int_{-1}^{2} \left((2 - x^2) - (-x)\right)\ dx = \int_{-1}^{2} \left(2 + x - x^2\right)\ dx \\
&= \left[2x + \frac{1}{2}x^2 - \frac{1}{3}x^3\right]_{-1}^{2} \\
&= \left(2(2) + \frac{1}{2}(2)^2 - \frac{1}{3}(2)^3\right) - \left(2(-1) + \frac{1}{2}(-1)^2 - \frac{1}{3}(-1)^3\right) \\
&= \left(4 + 2 - \frac{8}{3}\right) - \left(-2 + \frac{1}{2} + \frac{1}{3}\right) = 4.5.
\end{aligned}$$

Thus the area bounded by the two curves is 4.5 square units.

(b) The figure below shows the graphs of two functions, $y = f(x)$ and $y = g(x)$. Write an expression that represents the area bounded between these two curves.



Solution: This is like the previous problem, in that

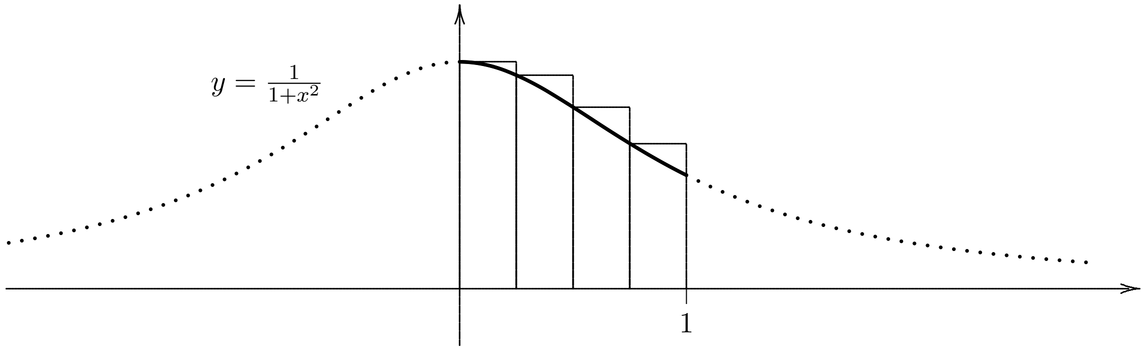
$$\text{Area bounded by the curves} = \int_a^b (\text{top curve} - \text{bottom curve})\ dx.$$

Here the added complication is that sometimes $f(x)$ is the top curve and sometimes $g(x)$ is. That is, the two curves bound a region from $x = 1$ to $x = 6$. From $x = 1$ to $x = 3$, $f(x)$ is the top curve and $g(x)$ is the bottom curve. On the other hand, from $x = 3$ to $x = 6$, $g(x)$ is the top curve and $f(x)$ is the bottom curve. Thus our final answer will have two integrals:

$$\text{Area bounded by the curves} = \int_1^3 (f(x) - g(x))\ dx + \int_3^6 (g(x) - f(x))\ dx.$$

This is the required expression.

6 (a) Approximate the integral $\int_0^1 \frac{1}{1+x^2}\,dx$ by partitioning the interval $[0,1]$ into four subintervals of equal length and choosing u as the left endpoint of each subinterval. Here is a graph of the function over the appropriate interval:



Solution: I've drawn some boxes on the graph, above. What we're doing here is approximating the area underneath the curve from $x = 0$ to $x = 1$ by the area of these four boxes. Each box has width

$$\Delta x = \frac{b-a}{n} = \frac{1-0}{4} = 0.25$$

and height $f(u)$. In each subinterval, we choose u as the left endpoint. Thus in $[0, 0.25]$, $u_1 = 0$; in $[0.25, 0.5]$, $u_2 = 0.25$; in $[0.5, 0.75]$, $u_3 = 0.75$; and in $[0.75, 1]$, $u_4 = 0.75$. The area of these four boxes is therefore

$$\begin{aligned}
\int_0^1 \frac{1}{1+x^2}\,dx = \text{Area} &\approx f(u_1)\Delta x + f(u_2)\Delta x + f(u_3)\Delta x + f(u_4)\Delta x \\
&= \Big(f(u_1) + f(u_2) + f(u_3) + f(u_4)\Big)\Delta x \\
&= \Big(f(0) + f(0.25) + f(0.5) + f(0.75)\Big)(0.25) \\
&= \left(\frac{1}{1+0^2} + \frac{1}{1+(0.25)^2} + \frac{1}{1+(0.5)^2} + \frac{1}{1+(0.75)^2}\right)(0.25) \\
&= \left(1 + \frac{16}{17} + \frac{4}{5} + \frac{16}{25}\right)(0.25) \\
&\approx 0.8453.
\end{aligned}$$

(b) Is your estimate in part (a) higher or lower than the actual value of the integral? Explain how you know this without knowing the actual value of the integral.

Solution: We have overestimated the integral. You can see this by noting that the integral is the area bounded by the curve $y = \frac{1}{1+x^2}$ and the x-axis; the four rectangles shown more than cover this area. Thus our estimate is too high. (The *actual* value of the integral is $\frac{\pi}{4} \approx 0.785398163$, so we have overestimated by about 0.06.)