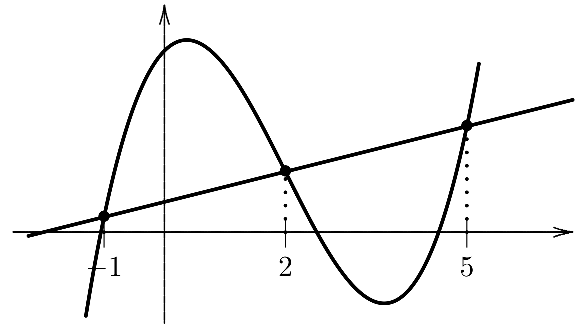
1 Find the area bounded by the curves $y = x+2$ and $y = x^3 - 6x^2 + 4x + 12$. The curves are graphed to the right. You may assume that the two curves intersect when $x = -1$, when $x = 2$, and when $x = 5$.



Solution: The area between two curves from $x = a$ to $x = b$ is, remember,

$$\text{Area} = \int_a^b \big(\text{ top curve } - \text{ bottom curve }\big)\, dx.$$

Here the cubic is the top curve from $x = -1$ to $x = 2$, but the line is the top curve from $x = 2$ to $x = 5$. Thus we must integrate in two pieces:

$$\begin{aligned}\text{Area} &= \int_{-1}^{2} \big((x^3 - 6x^2 + 4x + 12) - (x+2)\big)\, dx + \int_2^5 \big((x+2) - (x^3 - 6x^2 + 4x + 12)\big)\, dx \\ &= \int_{-1}^{2} \big(x^3 - 6x^2 + 3x + 10\big)\, dx + \int_2^5 \big(-x^3 + 6x^2 - 3x - 10\big)\, dx.\end{aligned}$$

We perform this integration:

$$\begin{aligned}\text{Area} &= \left(\frac{1}{4}x^4 - 2x^3 + \frac{3}{2}x^2 + 10x\right)\Bigg|_{-1}^{2} + \left(-\frac{1}{4}x^4 + 2x^3 - \frac{3}{2}x^2 - 10x\right)\Bigg|_{2}^{5} \\ &= \left(\frac{1}{4}(2)^4 - 2(2)^3 + \frac{3}{2}(2)^2 + 10(2)\right) - \left(\frac{1}{4}(-1)^4 - 2(-1)^3 + \frac{3}{2}(-1)^2 + 10(-1)\right) \\ &\qquad + \left(-\frac{1}{4}(5)^4 + 2(5)^3 - \frac{3}{2}(5)^2 - 10(5)\right) - \left(-\frac{1}{4}(2)^4 + 2(2)^3 - \frac{3}{2}(2)^2 - 10(2)\right) \\ &= (4 - 16 + 6 + 20) - \left(\frac{1}{4} + 2 + \frac{3}{2} - 10\right) + \left(-\frac{625}{4} + 250 - \frac{75}{2} - 50\right) - (-4 + 16 - 6 - 20) \\ &= 14 - \left(-\frac{25}{4}\right) + \frac{25}{4} - (-14) \\ &= \frac{81}{2} = 40.5.\end{aligned}$$

This is our answer.

2 Let X be a continuous random variable on $0 \le x \le 2$, with probability density function $f(x) = kx^2$.

(a) Compute the value of k that makes $f(x) = kx^2$ a probability density function.

Solution: Recall that a function $f(x)$ on a domain $a \le x \le b$ is a probability density function if $f(x) \ge 0$ and the integral of $f(x)$ from $x = a$ to $x = b$ is 1. The first fact simply says that $k \ge 0$; the key fact is the second one, that the integral of the probability density function must be 1. That is,

$$\int_0^2 kx^2 \, dx = 1.$$

This will allow us to solve for k:

$$1 = \int_0^2 kx^2 \, dx = k\frac{x^3}{3}\bigg|_0^2 = k\left(\frac{2^3}{3} - \frac{0^3}{3}\right) = k \cdot \frac{8}{3}.$$

Thus $k = 3/8 = 0.375$.

(b) Compute $P(1 \le X \le 1.5)$

Solution: This probability is the definite integral of the probability density function from $x = 1$ to $x = 1.5$:

$$P(1 \le X \le 1.5) = \int_1^{1.5} \frac{3}{8}x^2 \, dx = \frac{3}{8} \cdot \frac{x^3}{3}\bigg|_1^{1.5} = \frac{3}{8}\left(\frac{1.5^3}{3} - \frac{1^3}{3}\right) = \frac{19}{64} = 0.296875.$$

Thus this probability is about 29.69%.

(c) Compute $P(X \le 1)$

Solution: The difference between this part and part (b) is that there is no explicit lower limit given in $P(X \le 1)$. Since our probability density function $f(x)$ is defined on $0 \le X \le 2$, this probability is the same as $P(X \le 1) = P(0 \le X \le 1)$. Now this is the same sort of computation as in part (b):

$$P(0 \le X \le 1) = \int_0^1 \frac{3}{8}x^2 \, dx = \frac{3}{8} \cdot \frac{x^3}{3}\bigg|_0^1 = \frac{3}{8}\left(\frac{1^3}{3} - \frac{0^3}{3}\right) = \frac{1}{8} = 0.125.$$

Thus $P(X \le 1) = 12.5\%$.

(d) Compute $E(X)$, the expected value of X.

Solution: Recall that the expected value $E(X)$ for a continuous random variable X with probability density function $f(x)$ on $a \le x \le b$ is

$$E(X) = \int_a^b xf(x) \, dx.$$

Thus, in our case, the expected value is

$$E(X) = \int_0^2 x \cdot \frac{3}{8}x^2 \, dx = \frac{3}{8}\int_0^2 x^3 \, dx = \frac{3}{8}\frac{x^4}{4}\bigg|_0^2 = \frac{3}{8}\left(\frac{2^4}{4} - \frac{0^4}{4}\right) = \frac{3}{2} = 1.5.$$

3 Doris is waiting for a Circle Link shuttle bus. She's read on the web site that the buses run every 23 minutes. She decides to model the amount of time she must wait as random variable X with a uniform distribution taking values from 0 to 23 minutes.

(a) How long should Doris expect to wait for a bus? That is, what is the expected value of X?

Solution: Recall that the expected value $E(X)$ for a continuous random variable X with probability density function $f(x)$ on $a \leq x \leq b$ is

$$E(X) = \int_a^b x f(x)\, dx.$$

A uniform distribution has probability density function $f(x) = \frac{1}{b-a}$. Here the interval $a \leq x \leq b$ is from $a = 0$ to $b = 23$ minutes. Thus $f(x) = \frac{1}{23-0} = \frac{1}{23}$, so the expected value is

$$E(X) = \int_0^{23} x \cdot \frac{1}{23}\, dx = \frac{1}{23}\int_0^{23} x\, dx = \frac{1}{23}\frac{x^2}{2}\bigg|_0^{23} = \frac{1}{23}\left(\frac{23^2}{2} - \frac{0^2}{2}\right) = \frac{23}{2} = 11.5.$$

Thus Doris's expected wait is 11.5 minutes.

(b) What is the probability that Doris must wait 20 minutes or more for a bus?

Solution: This is $P(X \geq 20)$ or, equivalently, $P(20 \leq X \leq 23)$ (since $f(x)$ is defined on $0 \leq X \leq 23$). Thus, as in the previous problem,

$$P(20 \leq X \leq 23) = \int_{20}^{23} \frac{1}{23}\, dx = \frac{1}{23}x\bigg|_{20}^{23} = \frac{1}{23}(23 - 20) = \frac{3}{23} \approx 0.1304.$$

Thus roughly 13.04% of the time, Doris will have to wait at least 20 minutes for the bus.

(c) Doris is running late and will only be on time for her exam if the bus arrives within 3 minutes. What is the probability that a bus will arrive in 3 minutes or less?

Solution: This is $P(X \leq 3)$ or, equivalently, $P(0 \leq X \leq 3)$ (since, as in part (b), $f(x)$ is defined on $0 \leq X \leq 23$). Thus, as in the previous part,

$$P(0 \leq X \leq 3) = \int_0^3 \frac{1}{23}\, dx = \frac{1}{23}x\bigg|_0^3 = \frac{1}{23}(3 - 0) = \frac{3}{23} \approx 0.1304.$$

Thus roughly 13.04% of the time, Doris will have to wait at most 3 minutes for the bus.

4 A mechanic notices something about the cars that she works on. The mileage on the various cars is approximately normally distributed with mean 80 thousand miles and standard deviation 20 thousand miles.

(a) What is the probability that a randomly selected car (under the mechanic's care) has between 60 and 100 thousand miles on the odometer?

Solution: If X is the normally distributed random variable of mileage on the cars, with mean $\mu = 80$ thousand miles and standard deviation $\sigma = 20$ thousand miles, then we're asked for the probability $P(60 \leq X \leq 100$. As is usual with normal random variables, we write this probability in terms of the standard normal distribution Z (with mean $\mu = 0$ and standard deviation $\sigma = 1$):

$$P(60 \leq X \leq 100) = P\left(\frac{60-80}{20} \leq Z \leq \frac{100-80}{20}\right) = P\left(-1 \leq Z \leq 1\right).$$

By the symmetry of the normal distribution, this is

$$P(60 \leq X \leq 100) = P\left(-1 \leq Z \leq 1\right) = P\left(-1 \leq Z \leq 0\right) + P\left(0 \leq Z \leq 1\right) = 2P\left(0 \leq Z \leq 1\right).$$

Now we look this value up on the table and find that

$$P(60 \leq X \leq 100) = 2P\left(0 \leq Z \leq 1\right) \approx 2\left(0.3413\right) = 0.6826.$$

That is, the probability that a car has between 60 and 100 thousand miles is roughly 68.26%.

(b) If we say that a car is new if it has less than 15 thousand miles on it, then what percentage of the cars this mechanic works on are new cars?

Solution: This question asks for $P(X < 15)$, which as before we convert to a question about the standard normal distribution Z:

$$P(X < 15) = P\left(Z < \frac{15-80}{20}\right) = P\left(Z < -3.25\right).$$

By symmetry, this can be written as

$$\begin{aligned} P(X < 15) = P\left(Z < -3.25\right) &= P\left(Z > 3.25\right) \\ &= 1 - P(Z \leq 3.25) \\ &= 1 - P(Z < 0) - P(0 \leq Z \leq 3.25) \\ &= 0.5 - P(0 \leq Z \leq 3.25). \end{aligned}$$

This we can look up on the table: $P(X < 15) \approx 0.5 - 0.4994 = 0.0006$, or roughly 0.06%.

(c) The mechanic overhears someone say that 90% of the cars he works on have less than 150 thousand miles on them. She'd like to compute a similar number – she'd like to be able to say that 90% of the cars she works on have less than x thousand miles. What is x?

Solution: Now the question becomes, for what value of x is $P(X < x) = 0.9$? Again, we translate this into the standard normal distribution to find $P(X < x) = P\left(Z < \frac{x-80}{20}\right) = P(Z < z)$, where $z = \frac{x-80}{20}$. So for what value of z is $P(Z < z) = 0.9$? The table we're given shows $P(0 \leq Z < z)$, and since $P(Z < 0) = 0.5$ (by symmetry), we want to find z with $P(0 \leq Z < z) = 0.4$ (since $P(Z < z) = P(Z < 0) + P(0 \leq Z < z)$). To find this, we look at our table. Alas, nothing on the table gives us a Z-value of precisely 0.4000. We can interpolate, however, to find such a z:

z	1.28	z	1.29
Z-value	0.3997	0.4000	0.4015

Often what's done in this situation is we interpolate between 1.28 and 1.29 to find the z with Z-value 0.4000:

$$\frac{\Delta z}{\Delta\ \text{Value}} = \frac{z - 1.28}{0.4000 - 0.3997} = \frac{1.29 - 1.28}{0.4015 - 0.3997} \qquad \text{or} \qquad \frac{z - 1.28}{0.0003} = \frac{0.01}{0.0018}.$$

Thus $z = 1.28 + \frac{0.01}{0.0018}(0.0003) \approx 1.2817$. Solving for x in $z = \frac{x-80}{20}$, we get the following possible values:

z	x
1.28	105.6
1.2817	105.63
1.29	105.8

Thus we get an answer of around 105.6 thousand miles, and (roughly speaking) the answer is always around 106 thousand miles.

5 In parts (a) and (b), compute the integrals using any methods from the course:

(a) $\int x \cos(6x)\, dx$

Solution: Here we begin by making the substitution $t = 6x$. This is not strictly necessary (this will end up being an integration by parts problem), but it makes things a little cleaner. Then $x = t/6$, and $dx = \frac{1}{6}\, dt$. Thus

$$\int x \cos(6x)\, dx = \int \frac{t}{6} \cos(t) \cdot \frac{1}{6}\, dt = \frac{1}{36} \int t \cos(t)\, dt.$$

Now we integrate this using the technique of integration by parts. That is, we let $u = t$ and $dv = \cos(t)\, dt$, so $du = dt$ and $v = \sin(t)$. (Here u is chosen as the "problem" that is made better by differentiation.) Using the integration by parts formula

$$\int u\, dv = uv - \int v\, du,$$

Thus we get (making sure we keep track of that pesky $1/36$)

$$\begin{aligned}
\int x \cos(6x)\, dx &= \frac{1}{36} \int t \cos(t)\, dt \\
&= \frac{1}{36} \left(t \sin(t) - \int \sin(t)\, dt \right) \\
&= \frac{1}{36} \Big[t \sin(t) - (-\cos(t)) + K \Big] \\
&= \frac{1}{36} \big(t \sin(t) + \cos(t) + K \big) \\
&= \frac{1}{36} \big(6x \sin(6x) + \cos(6x) + K \big) \\
&= \frac{1}{6} x \sin(6x) + \frac{1}{36} \cos(6x) + C,
\end{aligned}$$

where we've replaced the constant $K/36$ with the constant C.

(b) $\int x^3 \sqrt{1 + x^2}\, dx$

Solution: This integral requires a substitution. Probably the simplest is simply $u = 1 + x^2$, so $du = 2x\, dx$ or $dx = \frac{1}{2x}\, du$. Thus

$$\int x^3 \sqrt{1 + x^2}\, dx = \int x^3 \sqrt{u} \cdot \frac{1}{2x}\, du = \frac{1}{2} \int x^2 \sqrt{u}\, du.$$

We haven't completely substituted u for x – this is usually a problem, but here we can solve for $x^2 = u - 1$. Thus

$$\int x^3 \sqrt{1 + x^2}\, dx = \frac{1}{2} \int x^2 \sqrt{u}\, du = \frac{1}{2} \int (u - 1)\, u^{1/2}\, du = \frac{1}{2} \int \left(u^{3/2} - u^{1/2} \right)\, du.$$

This is now an easy integral:

$$\begin{aligned}\int x^3\sqrt{1+x^2}\,dx = \frac{1}{2}\int\left(u^{3/2}-u^{1/2}\right)\,du &= \frac{1}{2}\left(\frac{1}{5/2}u^{5/2}-\frac{1}{3/2}u^{3/2}+K\right)\\ &= \frac{1}{2}\cdot\frac{2}{5}u^{5/2}-\frac{1}{2}\frac{2}{3}u^{3/2}+\frac{K}{2}\\ &= \frac{1}{5}u^{5/2}-\frac{1}{3}u^{3/2}+\frac{K}{2}\\ &= \frac{1}{5}\left(1+x^2\right)^{5/2}-\frac{1}{3}\left(1+x^2\right)^{3/2}+C,\end{aligned}$$

where we have replaced the constant $K/2$ with the constant C.

(c) Is it true that $\int_1^\infty \left(\frac{1}{x}\right)^2\,dx = \left(\int_1^\infty \frac{1}{x}\,dx\right)^2$? Fully (mathematically) justify your answer.

Solution: The two expressions are different. Let's compute each to see this.

First, we compute the integral on the left. This is an improper integral, so it is defined as a limit:

$$\begin{aligned}\int_1^\infty \left(\frac{1}{x}\right)^2\,dx &= \lim_{b\to\infty}\int_1^b x^{-2}\,dx\\ &= \lim_{b\to\infty}\left(\frac{1}{-1}x^{-1}\Big|_1^b\right) = \lim_{b\to\infty}\left(-\frac{1}{x}\Big|_1^b\right)\\ &= \lim_{b\to\infty}\left(-\frac{1}{b}-\left(-\frac{1}{1}\right)\right) = \lim_{b\to\infty}\left(-\frac{1}{b}+\frac{1}{1}\right)\\ &= -0+1=1.\end{aligned}$$

Thus the left-hand side is 1.

The right-hand side is the square of an improper integral that doesn't exist. We see this by computing the integral as a limit:

$$\int_1^\infty \frac{1}{x}\,dx = \lim_{b\to\infty}\int_1^b \frac{1}{x}\,dx = \lim_{b\to\infty}\ln(x)\Big|_1^b = \lim_{b\to\infty}\left(\ln(b)-\ln(1)\right) = \lim_{b\to\infty}\ln(b) = +\infty.$$

That is, the integral does not exist, and therefore its square does not equal 1 (the integral on the left-hand side).

6 Compute the inverse of each of the following matrices. If the inverse of a matrix does not exist, explain how you know this.

(a) $\begin{bmatrix} 1 & 4 \\ \frac{1}{2} & 2 \end{bmatrix}$

Solution: This matrix has no inverse. One way to find the inverse of a 2×2 matrix is by the formula

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}.$$

For our matrix, $ad - bc = 1 \cdot 2 - 4 \cdot \frac{1}{2} = 0$. This means that this matrix has no inverse.

The other way we know to find the inverse of a matrix A is to use elimination (row reduction) to turn the matrix $[\ A \mid I\ ]$ into $[\ I \mid A^1\ ]$. If this is not possible, then there is no inverse. We try to do this here:

$$\left[\begin{array}{cc|cc} 1 & 4 & 1 & 0 \\ \frac{1}{2} & 2 & 0 & 1 \end{array}\right] \longrightarrow \left[\begin{array}{cc|cc} 1 & 4 & 1 & 0 \\ 0 & 0 & -\frac{1}{2} & 1 \end{array}\right] \quad R_2 = r_2 - \tfrac{1}{2} r_1$$

Since there is a row of zeros to the left of the vertical line, there is no way we will be able to continue the elimination to get the identity matrix. (Notice that our elimination is reversible, so we can't possibly reduce to the identity matrix.)

(b) $\begin{bmatrix} 1 & 2 & -1 \\ 0 & 1 & 0 \\ -2\pi & -4\pi & 2\pi + 1 \end{bmatrix}$

Solution: For a 3×3 matrix, the only way we know to find inverses is by row reduction. Thus we'll proceed as in the second method in part (a):

$$\left[\begin{array}{ccc|ccc} 1 & 2 & -1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ -2\pi & -4\pi & 2\pi + 1 & 0 & 0 & 1 \end{array}\right] \longrightarrow \left[\begin{array}{ccc|ccc} 1 & 2 & -1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 2\pi & 0 & 1 \end{array}\right] \quad R_3 = r_3 + 2\pi r_1$$

$$\longrightarrow \left[\begin{array}{ccc|ccc} 1 & 0 & -1 & 1 & -2 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 2\pi & 0 & 1 \end{array}\right] \quad R_1 = r_1 - 2r_2$$

$$\longrightarrow \left[\begin{array}{ccc|ccc} 1 & 0 & -1 & 2\pi + 1 & -2 & 1 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 2\pi & 0 & 1 \end{array}\right] \quad R_1 = r_1 + r_3$$

Thus the required inverse matrix is $\begin{bmatrix} 2\pi + 1 & -2 & 1 \\ 0 & 1 & 0 \\ 2\pi & 0 & 1 \end{bmatrix}$.

7 Find all the values of α and β so that the system

$$\begin{array}{rcrcrcr} x & + & 3y & - & 2z & = & 5 \\ 2x & + & 4y & + & 3z & = & -3 \\ -x & + & y & + & \alpha z & = & \beta \end{array}$$

(a) . . . has a unique solution.

(b) . . . has no solution.

(c) . . . has infinitely many solutions.

Solution: To do this problem, we use elimination and see what happens. Let's try elimination (or row reduction):

$$\left[\begin{array}{rrr|r} 1 & 3 & -2 & 5 \\ 2 & 4 & 3 & -3 \\ -1 & 1 & \alpha & \beta \end{array}\right] \longrightarrow \left[\begin{array}{rrr|r} 1 & 3 & -2 & 5 \\ 0 & -2 & 7 & -13 \\ 0 & 4 & \alpha-2 & \beta+5 \end{array}\right] \begin{array}{l} R_2 = r_2 - 2r_1 \\ R_3 = r_3 + r_1 \end{array}$$

$$\longrightarrow \left[\begin{array}{rrr|r} 1 & 3 & -2 & 5 \\ 0 & -2 & 7 & -13 \\ 0 & 0 & \alpha+12 & \beta-21 \end{array}\right] \; R_3 = r_3 + 2r_2$$

$$\longrightarrow \left[\begin{array}{rrr|r} 1 & 3 & -2 & 5 \\ 0 & 1 & -3.5 & 6.5 \\ 0 & 0 & \alpha+12 & \beta-21 \end{array}\right] \; R_2 = \frac{1}{-2} r_2$$

Now we're stymied because we can no longer be sure we can divide by $\alpha + 12$. If we *could* divide by $\alpha + 12$, then we'd get something that looks like

$$\left[\begin{array}{rrr|r} 1 & 3 & -2 & 5 \\ 0 & 1 & -3.5 & 6.5 \\ 0 & 0 & 1 & \frac{\beta-21}{\alpha+12} \end{array}\right],$$

which has a unique solution. That is, if $\alpha \neq -12$, the system has a unique solution.

Now suppose $\alpha = -12$, so $\alpha + 12 = 0$ and our system row reduces to

$$\left[\begin{array}{rrr|r} 1 & 3 & -2 & 5 \\ 0 & 1 & -3.5 & 6.5 \\ 0 & 0 & 0 & \beta-21 \end{array}\right].$$

This last equation is $0x + 0y + 0z = \beta - 21$. Thus if $\beta - 21 \neq 0$, there are no solutions to this system. If, on the other, $\beta - 21 = 0$, we have simply

$$\left[\begin{array}{rrr|r} 1 & 3 & -2 & 5 \\ 0 & 1 & -3.5 & 6.5 \\ 0 & 0 & 0 & 0 \end{array}\right],$$

which has infinitely many solutions. Thus we have the following answers:

(a) There is a unique solution if $\alpha \neq -12$.

(b) There are no solutions if $\alpha = -12$ but $\beta \neq 21$.

(c) There are infinitely many solutions if $\alpha = -12$ and $\beta = 21$.

8 A population has a growth rate proportional to its size; that is,

$$\frac{dP}{dt} = kP.$$

Show how to derive the formula $P(t) = P_0 e^{kt}$ from this differential equation.

Solution: The differential equation is both a separable differential equation and a first-order linear differential equation. We will, naturally, solve this differential equation using the techniques for both types of equations.

First, we consider the equation as a separable differential equation. We solve this, naturally enough, by separating the two variables:

$$\frac{dP}{dt} = kP \qquad \text{becomes} \qquad \frac{dP}{P} = k\,dt.$$

Integrating both sides gives us

$$\ln(|P|) = kt + K.$$

Exponentiating both sides gives us

$$|P| = e^{\ln(|P|)} = e^{kt+K} = e^{kt} \cdot e^{K} \qquad \text{or} \qquad P = Ce^{kt}$$

where $C = \pm e^{K}$.

On the other hand, we can also treat the differential equation as a first-order linear differential equation. We write this in the standard form

$$P' - kP = 0$$

so the integrating factor is $h(t) = e^{\int (-k)\,dt} = e^{-kt}$. We multiply the equation by this factor to get

$$e^{-kt}P' - ke^{-kt}P = 0 \qquad \text{or} \qquad \left(e^{-kt}P\right)' = 0.$$

Integrating both sides gives us

$$e^{-kt}P = C \qquad \text{or} \qquad P = Ce^{kt},$$

as before.

Finally, we notice that $P(t) = Ce^{kt}$ has initial value $P(0) = Ce^{k \cdot 0} = Ce^{0} = C$. Thus C is the value of $P(t)$ at time $t = 0$, so we often call it P_0. That is, $P(t) = P_0 e^{kt}$.

9 Solve the differential equation $xy' + y = e^x$ with initial condition $y(1) = 2$.

Solution: This is a first-order linear differential equation, so we must put it in standard form

$$y' + P(x)y = G(x).$$

We do this by dividing the entire equation by x:

$$y' + \frac{1}{x}\,y = \frac{e^x}{x}.$$

Now we multiply through by an integrating factor $h(x) = e^{\int P(x)\,dx}$. Here $P(x) = {}^1\!/_x$, so the integrating factor is

$$h(x) = e^{\int P(x)\,dx} = e^{\int \frac{1}{x}\,dx} = e^{\ln(|x|)} = x.$$

(Here we've dropped the absolute value sign. Since the initial condition is $y = 2$ when $x = 1$, we're going to solve this equation for $x > 0$. Notice that $x = 0$ causes a problem in out "divide by x" step.) We multiply through by $h(x) = x$ to get

$$x\left(y' + \frac{1}{x}\,y = \frac{e^x}{x}\right) \qquad \text{or} \qquad xy' + y = e^x \qquad \text{or} \qquad (xy)' = e^x.$$

Integrating both sides of this last equation gives us $xy = e^x + K$, or

$$y = \frac{e^x}{x} + \frac{K}{x}.$$

Now we plug in our initial condition ($y = 2$ when $x = 1$) to find $K = 2 - e$. Thus

$$y = \frac{e^x}{x} + \frac{2-e}{x}.$$

10 A cup of coffee is purchased from McDonald's and brought to the exam, where the room temperature is 70° F. A quick measurement determines that the coffee is 180° F. We let the coffee cool for 5 minutes and find that now the coffee temperature is 175° F. Assume that the rate at which the coffee cools is proportional to the difference between the coffee temperature and the room temperature.

(a) Write down the initial value problem describing this situation. That is, write down a differential equation and an initial condition.

Solution: The last sentence of the description tells us how to write down the differential equation. If we let $T(t)$ be the temperature of the coffee after t minutes, then

$$T'(t) = k\,(T - 70)\,.$$

(Here $T'(t)$ is the "rate at which the coffee cools" and $T - 70$ is "the difference between the coffee temperature and the room temperature." The number k is the constant of proportionality.) The initial condition is $T(0) = 180$ (that is, the temperature is 180° F when $t = 0$ minutes). There is another condition, $T(5) = 175$, that we will use in part (b) to find the constant of proportionality k.

(b) Find a formula for the coffee temperature as a function of time. Do this by solving the initial value problem in part (a).

Solution: The differential equation we found in part (a) is both a separable differential equation and a first-order linear differential equation. We will, naturally, solve the equation by both methods.

First, we treat the differential equation as a separable differential equation. Write $T'(t)$ as $\frac{dT}{dt}$ to get

$$\frac{dT}{dt} = k(T - 70) \qquad \text{or} \qquad \frac{dT}{T - 70} = k\,dt.$$

We integrate this to get $\ln(|T - 70|) = kt + K$. Exponentiate both sides and we get $e^{\ln(|T-70|)} = e^{kt+K} = e^{kt} \cdot e^{K}$ or $T - 70 = Ce^{kt}$. This gives us $T = 70 + Ce^{kt}$.

Before we continue, we'll now solve the differential equation as a first-order linear differential equation. First we put it in the standard form

$$y' + P(x)y = G(x) \qquad \text{or} \qquad T' - kT = -70k.$$

The integrating factor for this equation is $h(t) = e^{\int P(t)\,dt} = e^{\int (-k)\,dt} = e^{-kt}$, so after multiplying our equation by this, we get

$$e^{-kt}T' - ke^{-kt}T = -70ke^{-kt} \qquad \text{or} \qquad \left(e^{-kt}T\right)' = -70ke^{-kt}.$$

Now integrating gives us $e^{-kt}T = 70e^{-kt} + C$, or (after multiplying through by e^{kt}), $T = 70 + Ce^{kt}$, as before.

Now we plug in the initial condition ($T = 180$ when $t = 0$) to find that $180 = 70 + Ce^{k\cdot 0} = 70 + C$, or $C = 110$. Thus $T(t) = 70 + 110e^{kt}$. We find the constant k by plugging in the other condition we noted in part (a) (that is, that $T = 175$ when $t = 5$). We find $175 = 70 + 110e^{k\cdot 5}$, or $e^{5k} = \frac{105}{110} = \frac{21}{22}$. Taking the natural log of both sides lets us find that $k = \frac{1}{5}\ln(\frac{21}{22}) \approx -0.009304$. Thus

$$T(t) = 70 + 110e^{\frac{1}{5}\ln\left(\frac{21}{22}\right)t} \approx 70 + 110e^{-0.009304t}.$$

(c) We'd like to drink the coffee when it is 135° F. How long will it take for the coffee to cool to this temperature?

Solution: Here we'd like to solve the equation $T(t) = 135$ for the time t. From part (b), this equation is

$$135 = 70 + 110e^{\frac{1}{5}\ln\left(\frac{21}{22}\right)t} \qquad \text{or} \qquad e^{\frac{1}{5}\ln\left(\frac{21}{22}\right)t} = \frac{65}{110} = \frac{13}{22}.$$

Taking the natural log and solving allows us to find that

$$t = t\,\frac{\ln\left({}^{13}\!/_{22}\right)}{\ln\left({}^{21}\!/_{22}\right)} \approx 56.5448 \text{ minutes.}$$

Thus the coffee will be 135° F in about 56.54 minutes.

11 The following table relates the average daily temperature at different portions of a creek with the elevation of that portion of the creek above sea level.

Elevation (kilometers):	2.7	2.8	3.2	3.5
Average Temperature (degrees Celsius):	11.2	10	8.5	7.5

(a) Show how to find a line of best fit using the method of least squares for the data in the table above.

Solution: There are, as usual, two approaches to finding the best fit line $y = mx + b$. The first is the linear algebra approach, which involves looking at the system $A^T AX = A^T Y$, where

$$A = \begin{bmatrix} x_1 & 1 \\ x_2 & 1 \\ \vdots & \vdots \\ x_n & 1 \end{bmatrix} = \begin{bmatrix} 2.7 & 1 \\ 2.8 & 1 \\ 3.2 & 1 \\ 3.5 & 1 \end{bmatrix}, \qquad X = \begin{bmatrix} m \\ b \end{bmatrix}, \qquad \text{and} \qquad Y = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix} = \begin{bmatrix} 11.2 \\ 10 \\ 8.5 \\ 7.5 \end{bmatrix}.$$

Then the linear system $A^T AX = A^T Y$ is

$$\begin{bmatrix} 2.7 & 2.8 & 3.2 & 3.5 \\ 1 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 2.7 & 1 \\ 2.8 & 1 \\ 3.2 & 1 \\ 3.5 & 1 \end{bmatrix} \begin{bmatrix} m \\ b \end{bmatrix} = \begin{bmatrix} 2.7 & 2.8 & 3.2 & 3.5 \\ 1 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 11.2 \\ 10 \\ 8.5 \\ 7.5 \end{bmatrix}.$$

or

$$\begin{bmatrix} 37.62 & 12.2 \\ 12.2 & 4 \end{bmatrix} \begin{bmatrix} m \\ b \end{bmatrix} = \begin{bmatrix} 111.69 \\ 37.2 \end{bmatrix} \qquad (*)$$

On the other hand, we could also use the partial derivatives approach, from which we've learned that the linear system to solve in order to find m and b is

$$\begin{bmatrix} \sum_{i=1}^{n} x_i^2 & \sum_{i=1}^{n} x_i \\ \sum_{i=1}^{n} x_i & n \end{bmatrix} \begin{bmatrix} m \\ b \end{bmatrix} = \begin{bmatrix} \sum_{i=1}^{n} x_i y_i \\ \sum_{i=1}^{n} y_i \end{bmatrix}.$$

Let's compute with the data:

i	x_i	y_i	x_i^2	$x_i y_i$
1	2.7	11.2	7.29	30.24
2	2.8	10	7.84	28
3	3.2	8.5	10.24	27.2
4	3.5	7.5	12.25	26.25
Sum:	12.2	37.2	37.62	111.69

We can then see the linear system is precisely that shown in equation (∗). We solve this system by using the inverse matrix to see that

$$\begin{bmatrix} m \\ b \end{bmatrix} = \begin{bmatrix} 37.62 & 12.2 \\ 12.2 & 4 \end{bmatrix}^{-1} \begin{bmatrix} 111.69 \\ 37.2 \end{bmatrix}$$

$$= \frac{1}{(37.62)(4) - (12.2)^2} \begin{bmatrix} 4 & -12.2 \\ -12.2 & 37.62 \end{bmatrix} \begin{bmatrix} 111.69 \\ 37.2 \end{bmatrix}$$

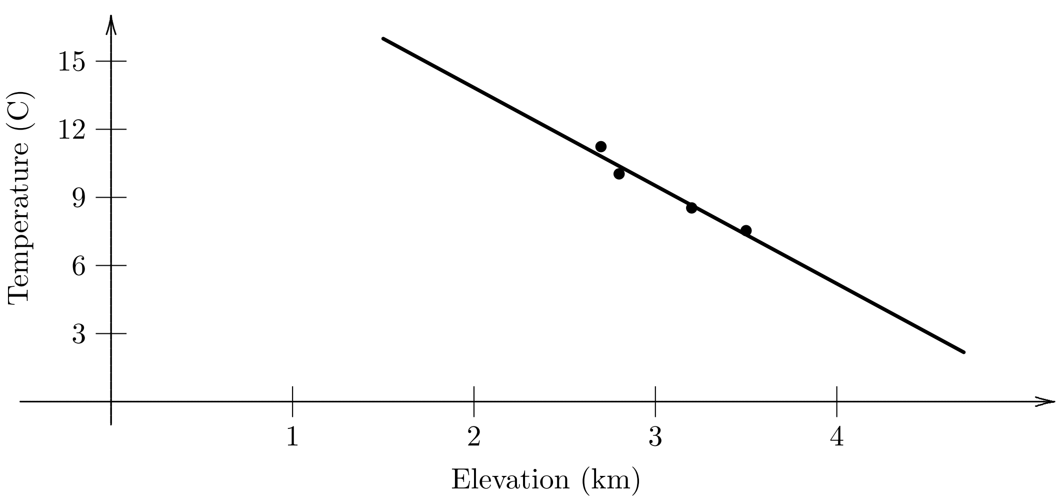
$$= \frac{1}{1.64} \begin{bmatrix} -7.08 \\ 36.846 \end{bmatrix}.$$

Thus our solution is $m = -\frac{7.08}{1.64} = -\frac{177}{41} \approx -4.3171$ and $b = \frac{36.846}{1.64} = \frac{18423}{820} \approx 22.4761$. That is, the least squares line is approximately $y = -4.3171x + 22.4761$.

(b) Use the equation you just found to compute the average daily temperature for this creek at an altitude of 4.2 kilometers.

Solution: This asks for us to simply find the average daily temperature y when the altitude is $x = 4.2$ kilometers. We simply plug in to the equation we found in part (a) to get $y \approx -4.3171(4.2) + 22.4761 \approx 4.3$ C. (Notice that an answer with four digits, like $y \approx 4.3354$, we need to keep more digits in b and, especially, m.)

The test was written with $x = 3.2$ instead of $x = 4.2$ kilometers. We changed this because $x = 3.2$ is a data point, and one gets that $y \approx 8.6524$ C when $x = 3.2$ kilometers. This is different than the data point ($y = 8.5$ when $x = 3.2$). Is this a problem? No, because the best fit line does not necessarily pass through every, or even *any*, data point. All we're saying is that the model we have says that the temperature should be around 8.65 C at an altitude of 3.2 kilometers, not that this is the actual temperature. The picture below shows the four given data points and the model line. As you can see, the line is *close* to the data points, but it does not pass through any of them. (For example, as we have seen at $x = 3.2$ kilometers, the data is a temperature of 8.5 C and line predicts a temperature of 8.65 C.)



12 Let $f(x, y) = 3x^2y + y^3 - 3x^2 - 3y^2 + 2$.

(a) Find all the first and second partial derivatives f_x, f_y, f_{xx}, f_{yy}, f_{xy}, and f_{yx} of the function $f(x, y)$.

Solution: We compute the first derivatives. Recall that f_x is the derivative of $f(x, y)$ with respect to x while treating y as a constant. Thus

$$f_x = 6xy + 0 - 6x - 0 + 0,$$

or $f_x = 6xy - 6x$. Similarly, $f_y = 3x^2 + 3y^2 - 6y$.
Second derivatives are similar. Recall that $f_{xy} = (f_x)_y$, the partial derivative of f_x with respect to y. Thus

$$f_{xx} = 6y - 6 \qquad f_{xy} = 6x \qquad f_{yx} = 6x \qquad \text{and} \qquad f_{yy} = 6y - 6.$$

Recall that $f_{xy} = f_{yx}$, so this is a good check to see that we've done this correctly.

(b) Find all the critical points of the function $f(x, y)$.

Solution: The critical points of $f(x, y)$ are the common solutions of the two equations $f_x = 0$ and $f_y = 0$, or

$$6xy - 6x = 0 \qquad \text{and} \qquad 3x^2 + 3y^2 - 6y = 0.$$

The first equation factors into $6x(y - 1) = 0$, so $x = 0$ or $y = 1$. If $x = 0$, then the second equation tells us that $3y^2 - 6y = 0$, or $3y(y - 2) = 0$, so $y = 0$ or $y = 2$. This gives us two points: $(x, y) = (0, 0)$ or $(0, 2)$. On the other hand, if $y = 1$, then the second equation becomes $3x^2 + 3(1)^2 - 6(1) = 0$, or $x^2 = 1$. Thus $x = \pm 1$, so we have two more points: $(x, y) = (1, 1)$ and $(-1, 1)$.

(c) Classify each of the critical points you found in part (b) as a local maximum, a local minimum, or a saddle point.

Solution: Recall that we use the discriminant

$$D = f_xxf_yy - f_{xy}^2 = (6y - 6)^2 - (6x)^2 = 36\left(-x^2 + y^2 - 2y + 1\right)$$

to classify each critical point as a local maximum, a local minimum, or a saddle point. We test each point in turn.

$(\mathbf{x}, \mathbf{y}) = (\mathbf{0}, \mathbf{0})$: Here $D = 36 > 0$, so the critical point is either a local maximum or a local minimum. Since $f_{xx} = 6(0) - 6 < 0$, this point is a local maximum.

$(\mathbf{x}, \mathbf{y}) = (\mathbf{0}, \mathbf{2})$: Here $D = 36 > 0$, so the critical point is either a local maximum or a local minimum. Since $f_{xx} = 6(2) - 6 > 0$, this point is a local minimum.

$(\mathbf{x}, \mathbf{y}) = (\mathbf{1}, \mathbf{1})$: Here $D = -36 < 0$, so this critical point is a saddle point.

$(\mathbf{x}, \mathbf{y}) = (-\mathbf{1}, \mathbf{1})$: Here $D = -36 < 0$, so this critical point is a saddle point.

13 Find a point on the line $3x + 4y = 5$ that is closest to the point $(0, 0)$ using Lagrange multipliers. Find this minimum distance.
Hint: It is easier to work with the *square* of the distance.

Solution: We wish to minimize the distance d between the origin $(0, 0)$ and the point (x, y), where this point (x, y) is constrained to lie on the line $3x + 4y = 5$. This distance is

$$d = \sqrt{(x-0)^2 + (y-0)^2} = \sqrt{x^2 + y^2}.$$

Using the hint, we will try to minimize

$$f(x, y) = d^2 = x^2 + y^2,$$

the square of the distance, rather than the distance itself. Our constraint is that (x, y) must lie on the given line, so $g(x, y) = 3x + 4y - 5 = 0$.

Now we proceed with the technique of Lagrange multipliers. That is, we form the new function

$$\begin{aligned} F(x, y, \lambda) &= f(x, y) + \lambda g(x, y) \\ &= x^2 + y^2 + \lambda\,(3x + 4y - 5)\,. \end{aligned}$$

The minimum of $f(x, y)$ subject to the constraint $g(x, y) = 0$ will be a critical point of this function F. Thus we solve the equations $F_x = 0$, $F_y = 0$, and $F_\lambda = 0$. We compute these derivatives:

$$F_x = 0 = 2x + 3\lambda \qquad F_x = 0 = 2y + 4\lambda \qquad \text{and} \qquad F_\lambda = 0 = 3x + 4y - 5.$$

From the first two equations, we find $\lambda = -\frac{2x}{3}$ and $\lambda = -\frac{y}{2}$. Setting these equal gives us $y = \frac{4x}{3}$. We plug this into the last equation (which is, as usual, our constraint equation):

$$0 = 3x + 4y - 5 = 3x + 4\left(\frac{4x}{3}\right) - 5 = 3x + \frac{16}{3}\,x - 5 = \frac{25}{3}\,x - 5,$$

from which we find $x = 3/5$. Since $y = \frac{4x}{3}$, we then get $y = 4/5$. Thus the point on the line $3x + 4y = 5$ closest to the origin is $\left(3/5, 4/5\right)$. The minimum distance from the origin to this line is the distance from the origin to this point $\left(3/5, 4/5\right)$, which is

$$d = \sqrt{\left(\frac{3}{5} - 0\right)^2 + \left(\frac{4}{5} - 0\right)^2} = \sqrt{\frac{9 + 16}{25}} = 1.$$