Normal Distributions	Standard Normal Distributions
Random Variable X	Random Variable Z
$f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}$	$f(z) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}z^2}$
mean $= E(X) = \mu$	mean $= E(Z) = 0$
standard deviation $= \sqrt{\mathrm{Var}(X)} = \sigma$	standard deviation $= \sqrt{\mathrm{Var}(Z)} = 1$

$$P(x_1 \le X \le x_2) = P\left(\frac{x_1 - \mu}{\sigma} \le Z \le \frac{x_2 - \mu}{\sigma}\right)$$

$P(0 \le Z \le z) = Z$-value (see tables)

1 Suppose an exam has average 70 and standard deviation 15, and the scores are assumed to be normally distributed. Find the probability that an exam, selected at random, has score...

(a) ... between 70 and 85.

(b) ... between 70 and 100.

(c) ... under 55.

(d) Find the cutoff score for the top 25% and the bottom 25%. That is, find the scores which 25% of the class did better than and which 25% of the class did worse than.

2 A study of births in Norway, 1992–1998, shows that birth weight is essentially normally distributed with mean $\mu = 3668$ g and standard deviation $\sigma = 511$ g. For our purposes we will assume that the birth weight is normally distributed.

Reference: `http://eb.niehs.nih.gov/bwt/subcfreq.htm`

(a) Using the conversion factor 1 pound is about 454 grams, find the probability that a baby weighs between 7 and 9 pounds at birth.

(b) What is the weight of the heaviest 10% of these Norwegian babies? That is, find a birth weight so that 90% of births are less than this weight and 10% are greater.

3 Suppose the life span of an automobile transmission is approximately normally distributed with mean $\mu = 30,000$ miles and standard deviation $\sigma = 3,000$ miles.

(a) Find the probability that a transmission lasts no more than $24,000$ miles.

(b) Find the probability that a transmission lasts at least $40,000$ miles.

(c) Suppose the manufacturer wants to guarantee their product by offering full refunds if the transmission fails within a certain number of miles. The company only wants to refund at most 5% of their sales. For how many miles should they warrant their transmissions?

1 (a) $P(70 \le X \le 85) = P\left(\frac{70-70}{15} \le Z \le \frac{85-70}{15}\right) = P(0 \le Z \le 1) = 0.3413$

(b) $P(70 \le X \le 100) = P\left(\frac{70-70}{15} \le Z \le \frac{100-70}{15}\right) = P(0 \le Z \le 2) = 0.4772$

(c) $P(X < 55) = P\left(Z < \frac{55-70}{15}\right) = P(Z < -1)$ Now the trick is to use symmetry to see that this is the same as $P(Z > 1)$. This we can't look up in the tables, but together with $P(0 \le Z \le 1)$ it adds to $P(Z \ge 0) = 1/2$. Thus

$$P(X < 55) = P(Z \ge 0) - P(0 \le Z \le 1) = 0.5000 - 0.3413 = 0.1587.$$

(d) We want to find x so that $P(X \ge x) = 0.25$. Equivalently, we can find $z = \frac{x-\mu}{\sigma}$ with $P(Z \ge z) = 0.25$. We can *almost* look up this z in our table:

z	Z-value
0.67	0.2486
z	0.2500
0.68	0.2517

Often what's done in this situation is we interpolate between 0.67 and 0.68 to find the z with Z-value 0.2500:

$$z \approx 0.67 + \frac{0.68-0.67}{0.2517-0.2486}(0.2500-0.2486) \approx 0.6745.$$

Solving for x in $z = \frac{x-70}{15}$, we get the following possible values:

z	x
0.67	80.05
0.6745	80.1175
0.68	80.2

As you can see, the answer is about 80 whether we interpolate or opt to use $z = 0.67$ or $z = 0.68$. Thus about 25% of the exam scores are over 80; by symmetry, about 25% are below 60 (10 points *below* the mean, just as 80 is 10 points above).

2 (a) We compute (rounding to the nearest gram). Note first that 7 pounds is 3178 grams (using our conversion factor – this is actually off by almost 50 grams) and 9 pounds is 4086 grams. Thus we want

$$\begin{aligned} P(3178 \le X \le 4086) &= P\left(\frac{3178-3668}{511} \le Z \le \frac{4086-3668}{511}\right) \\ &\approx P(-0.96 \le Z \le 0.82) \qquad \text{(after rounding)} \\ &= P(-0.96 \le Z \le 0) + P(0 \le Z \le 0.82) \\ &= P(0 \le Z \le 0.96) + P(0 \le Z \le 0.82) \qquad \text{(by symmetry)} \\ &= 0.3315 + 0.2939 = 0.6254. \end{aligned}$$

Thus roughly 62.5% of Norwegian babies weigh between 7 and 9 pounds at birth.

(b) We want to find a weight x so that $P(X \geq x) = 0.90$ or 90%. Since

$$P(X \geq x) = P\left(Z \geq \frac{x - 3668}{511}\right),$$

this is the same as finding a $z = \frac{x-3668}{511}$ for which $P(Z \geq z) = 0.90$, or (equivalently) $P(0 \leq Z \leq z) = 0.40$. Interpolating from our table, we find that $z \approx 1.282$, so $x \approx 4323$ grams. (If we hadn't interpolated, our answers might vary from 4322 to 4327, a spread of 5 grams, or about 0.011 pounds.)

3 (a) We're asked to find $P(X \leq 24,000)$. We convert this to a question about the standard normal distribution in the usual way:

$$\begin{aligned} P(X \leq 24,000) &= P\left(Z \leq \frac{24,000 - 30,000}{3,000}\right) \\ &= P(Z \leq -2). \end{aligned}$$

The standard normal distribution is symmetric about $z = 0$, so $P(Z \leq -2) = P(Z \geq 2)$. The table, of course, gives us probabilities like $P(0 \leq Z \leq 2)$. The key is that $P(Z \geq 0) = 0.5$ (by symmetry: the standard normal distribution is centered around 0, so the probability that it is positive is one-half). Thus

$$\begin{aligned} P(X \leq 2r,000) &= P(Z \leq -2) = P(Z \geq 2) \\ &= P(Z \geq 0) - P(0 \leq Z \leq 2) \\ &\approx 0.5000 - 0.4772 = 0.0228. \end{aligned}$$

That is, only about 2.28% of these transmissions last 24,000 miles or less.

(b) We do the usual computation:

$$\begin{aligned} P(X \geq 40,000) &= P\left(Z \geq \frac{40,000 - 30,000}{3,000}\right) \\ &\approx P(Z \geq 3.33) \\ &= P(Z \geq 0) - P(0 \leq Z \leq 3.33) \\ &\approx 0.5000 - 0.4996 \approx 0.0004. \end{aligned}$$

That is, only about 0.04% of these transmissions last 40,000 miles.

(c) We're asked to find x so that $P(X \leq x) = 0.05$ (or 5%). This is equivalent to

$$P\left(Z \leq z = \frac{x - 30,000}{3,000}\right) = 0.05$$

or $P(Z \geq |z|) = 0.05$ (by symmetry). The usual trick turns this into a Z-value:

$$P(0 \leq Z \leq |z|) = P(Z \geq 0) - P(Z \geq |z|) = 0.5 - 0.05 = 0.45.$$

Looking up a value of 0.4500 in our tables gives us $|z|$ lies between 1.64 and 1.65. We interpolate and find that $|z| \approx 1.645$, or $z \approx -1.645$. Thus x is about 25,065 miles. That is, the manufacturer should guarantee that their transmissions will last about 25,000 miles.