1 Consider a population P satisfying the differential equation $\dfrac{dP}{dt} = 5P^2 - 200P$.

(a) Identify the equilibrium solutions of the differential equation. (That is, for what values of P will the population be constant?)

Solution: If the population is constant (that is, if $P =$ constant), then $\frac{dP}{dt} = 0$ (the derivative of a constant is zero). We plug this into the differential equation and find that

$$0 = \frac{dP}{dt} = 5P^2 - 200P = 5P(P - 40),$$

so if P is constant then it must be either $P = 0$ or $P = 40$. These are the equilibrium solutions of our differential equation.

(b) For what values of P is the population increasing?

Solution: Another way to phrase this question is: for what values of P is $\frac{dP}{dt} > 0$? Since $\frac{dP}{dt} = 5P(P - 40)$, we can see that $\frac{dP}{dt} > 0$ when $P > 40$ (so both $P > 0$ and $P - 40 > 0$) and when $P < 0$ (so both $P < 0$ and $P - 40 < 0$ and the product is positive). For $0 < P < 40$, the terms are opposite sign ($P > 0$ but $P - 40 < 0$), so the product (and thus $\frac{dP}{dt}$) is negative.

We can display this computation in a handy chart:

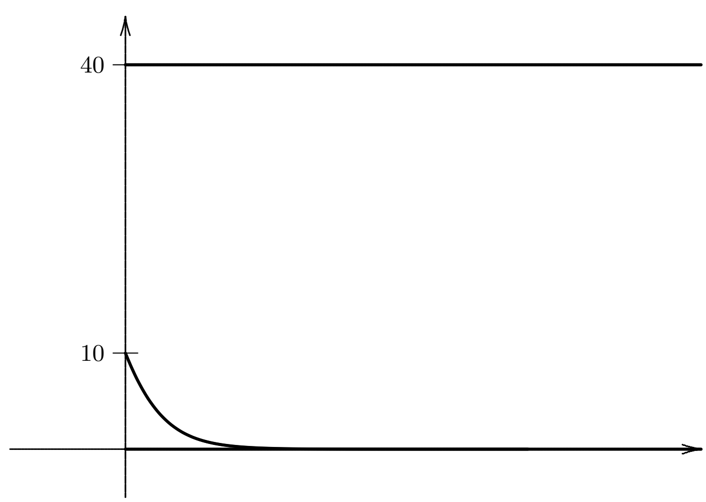
P		0		40	
sign of P	$-$	0	$+$	$+$	$+$
sign of $P - 40$	$-$	$-$	$-$	0	$+$
sign of $\frac{dP}{dt}$	$+$	0	$-$	0	$+$

The chart shows what we claimed: that $\frac{dP}{dt} = 0$ at $P = 0$ and at $P = 40$, and that P is increasing (that is, that $\frac{dP}{dt} > 0$) for $P < 0$ and $P > 40$.

(c) Describe what happens to the population as $t \to \infty$ if the initial population is $P(0) = 10$.

Solution: If the population starts at $P = 10$, then the population is decreasing (see part (b)). It continues to decrease provided the population is positive. But there is a constant solution $P = 0$ that our solution can never cross (by the uniqueness theorem of solutions to differential equations). Thus our solution decreases toward zero but always remains positive.

Here is a graph of three solutions: the two constant solutions ($P = 0$ and $P = 40$) as well as the solution with initial population $P(0) = 10$).



While it looks like the solution curves cross, in reality the curve starting at $P(0) = 10$ asymptotically approaches zero.

2 A large tank is filled to capacity with 500 gallons of pure water. Brine containing 2 pounds of salt per gallon is pumped into the tank at a rate of 5 gal/min. The well-mixed solution is pumped out at the same rate.

(a) Find an expression for the amount of salt in the tank after t minutes.

Solution: We solve this by writing down a differential equation involving $A(t)$, the amount of salt at time t, and its derivative. The derivative $A'(t)$ is the rate of change of the amount of salt. This rate of change involves two components:

- Salt is flowing *into* the vat in a liquid which comes in at 5 gallons per minute, with 2 pounds of salt per gallon.
- Salt is flowing *out* of the vat, again at 5 gallons per minute, but now the concentration is the concentration of the entire mixture. This is $A(t)$ pounds of salt per 500 gallons of liquid, or $\frac{A(t)}{500}$ pounds per gallon.

Thus our differential equation is

$$\frac{dA}{dt} = + \left(5\ \frac{\text{gal}}{\text{min}}\right)\left(2\frac{\text{lb}}{\text{gal}}\right) - \left(5\ \frac{\text{gal}}{\text{min}}\right)\left(\frac{A(t)\ \text{lb}}{500\ \text{gal}}\right)$$

or

$$A' = 10 - \frac{1}{100}A.$$

The initial condition is $A(0) = 0$: at time $t = 0$, there is no salt ("pure water") in the vat.

Now we solve the initial value problem we found above. This differential equation is both separable and first-order linear, so naturally we solve it using both techniques.

As a Separable Equation: We begin by treating the differential equation as separable. We write A' as $\frac{dA}{dt}$ and factor the $\frac{1}{100}$ from the right-hand side to write this as

$$\frac{dA}{dt} = \frac{1}{100}(1000 - A) = -\frac{1}{100}(A - 1000)$$

(where we've subsequently factored a -1 out of the right to make the coming integration simpler). We multiply by dt and divide by $A - 1000$ to obtain

$$\frac{dA}{A-1000} = -\frac{1}{100}\,dt \qquad \text{or} \qquad \int \frac{dA}{A-1000} = -\int \frac{1}{100}\,dt.$$

After integrating, we get $\ln(|A - 1000|) = -\frac{t}{100} + K$. Exponentiating, this becomes $|A - 1000| = e^{-t/100+K} = e^{-t/100} \cdot e^K$. Writing C for $\pm e^K$, our equation is $A - 1000 = Ce^{-t/100}$ or $A(t) = 1000 + Ce^{-t/100}$.

Now we plug in our initial condition $A(0) = 0$ to find that $0 = 1000 + Ce^0 = 1000 + C$, so $C = -1000$ and $A(t) = 1000 - 1000e^{-t/100}$. This is our answer.

As a First-Order Linear Equation: Now we treat the differential equation as first-order linear. We write the equation in the standard form $y' + P(t)y = G(t)$ to get

$$A' + \frac{1}{100}A = 10.$$

The integrating factor is $h(t) = e^{\int P(t)\,dt}$. Here $P(t) = \frac{1}{100}$, so $\int P(t)\,dt = \frac{t}{100}$ (we ignore any arbitrary constants) and so $h(t) = e^{t/100}$. Multiplying our differential equation by this integrating factor, we get

$$e^{t/100}A' + \frac{1}{100}e^{t/100}A = 10e^{t/100} \qquad \text{or} \qquad \left(e^{t/100}A\right)' = 10e^{t/100}.$$

(The left-hand side becomes the derivative of $h(t)y$; this is the reason we choose $h(t)$ in the way that we do.) Now we integrate:

$$\int \left(e^{t/100}A\right)'\,dt = \int 10e^{t/100}\,dt \qquad \text{or} \qquad e^{t/100}A(t) = 10 \cdot \frac{1}{1/100}e^{t/100} + K = 1000e^{t/100} + K.$$

If we multiply through by $e^{-t/100}$, we get $A(t) = 1000 + Ke^{-t/100}$, as before. Now using the initial condition $A(0) = 0$, we find that $K = -1000$ (as before), so $A(t) = 1000 - 1000e^{-t/100}$. This is the same answer we obtained earlier.

(b) In the long run, how much salt should we expect to have in the vat?

Solution: One way to do this is to take the limit of the answer from part (a):

$$\text{Limiting amount of salt} = \lim_{t\to\infty} A(t) = \lim_{t\to\infty}\left(1000 - 1000e^{-t/100}\right) = 1000 - 1000\cdot 0 = 1000 \text{ pounds.}$$

Another approach is to realize that the limiting concentration should be the same as the concentration of the incoming liquid. This is 2 pounds per gallon, so in the long run we should expect our 500 gallon in the tank to approach a concentration of 2 pounds per gallon, for a total of 500 gallons $\times$ 2 $\frac{\text{pounds}}{\text{gallon}}$ = 1000 pounds, as before.

3 Let $A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & -1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$.

(a) Use elimination to find A^{-1}.

Solution: This means that we will row-reduce (that is, use elimination) the extra-augmented matrix $[\ A \mid I\ ]$ to the matrix $[\ I \mid A^{-1}\ ]$, from which we simply read off the inverse. We will put this matrix in reduced row-echelon form more or less without comment:

$$[\ A \mid I\ ] = \left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 1 & -1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right] \to \left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & -2 & 0 & -1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right] \quad R_2 = r_2 - r_1$$

$$\to \left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1/2 & -1/2 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right] \quad R_2 = -\tfrac{1}{2}r_2$$

$$\to \left[\begin{array}{ccc|ccc} 1 & 1 & 0 & 1 & 0 & -1 \\ 0 & 1 & 0 & 1/2 & -1/2 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right] \quad R_1 = r_1 - r_3$$

$$\to \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1/2 & 1/2 & -1 \\ 0 & 1 & 0 & 1/2 & -1/2 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right] \quad R_1 = r_1 - r_2$$

$$= [\ I \mid A^{-1}\ ].$$

Thus $A^{-1} = \begin{bmatrix} 1/2 & 1/2 & -1 \\ 1/2 & -1/2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$.

(b) Find constants a, b, and c so that the parabola given by $y = ax^2 + bx + c$ passes through the points $(1,1)$, $(-1,-5)$, and $(0,-4)$.

Solution: If we plug in the point $(x,y) = (1,1)$ into our desired parabola, we get the equation $1 = a(1)^2 + b(1) + c$, or $a + b + c = 1$. We'll do this three times to get the linear system

$$\begin{aligned} a + b + c &= 1 \\ a - b + c &= -5 \\ c &= -4. \end{aligned}$$

If we write this in matrix form, we get the linear system

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & -1 & 1 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 1 \\ -5 \\ -4 \end{bmatrix}.$$

Notice that the 3×3 matrix on the left-hand side is the matrix A from part (a). Thus we can solve this equation by using the inverse we already found:

$$\begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & -1 & 1 \\ 0 & 0 & 1 \end{bmatrix}^{-1}\begin{bmatrix} 1 \\ -5 \\ -4 \end{bmatrix} = \begin{bmatrix} 1/2 & 1/2 & -1 \\ 1/2 & -1/2 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ -5 \\ -4 \end{bmatrix} = \begin{bmatrix} \frac{1}{2}(1) + \frac{1}{2}(-5) - 1(-4) \\ \frac{1}{2}(1) - \frac{1}{2}(-5) + 0(-4) \\ 0(1) + 0(-5) + 1(-4) \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \\ -4 \end{bmatrix}.$$

Thus $(a,b,c) = (2,3,-4)$, so our parabola is $y = 2x^2 + 3x - 4$.

4 Consider the system of equations below:

$$\begin{array}{rcrcrcrcrcr} x_1 & + & 2x_2 & + & x_3 & + & 2x_4 & - & 3x_5 & = & -3 \\ x_1 & - & x_2 & + & 4x_3 & + & 2x_4 & + & 9x_5 & = & 6 \\ 2x_1 & + & x_2 & + & 5x_3 & + & 5x_4 & + & 3x_5 & = & 1 \end{array}$$

(a) Write the augmented matrix for the system of equations and use row elimination to put it in reduced row-echelon form. Show the details of your computation, not just the final answer.

Solution: The corresponding augmented matrix is

$$\left[\begin{array}{rrrrr|r} 1 & 2 & 1 & 2 & -3 & -3 \\ 1 & -1 & 4 & 2 & 9 & 6 \\ 2 & 1 & 5 & 5 & 3 & 1 \end{array}\right].$$

We put this matrix in reduced row-echelon form by performing the elimination, essentially without comment:

$$\left[\begin{array}{rrrrr|r} 1 & 2 & 1 & 2 & -3 & -3 \\ 1 & -1 & 4 & 2 & 9 & 6 \\ 2 & 1 & 5 & 5 & 3 & 1 \end{array}\right] \to \left[\begin{array}{rrrrr|r} 1 & 2 & 1 & 2 & -3 & -3 \\ 0 & -3 & 3 & 0 & 12 & 9 \\ 0 & -3 & 3 & 1 & 9 & 7 \end{array}\right] \quad \begin{array}{l} \\ R_2 = r_2 - r_1 \\ R_3 = r_3 - 2r_1 \end{array}$$

$$\to \left[\begin{array}{rrrrr|r} 1 & 2 & 1 & 2 & -3 & -3 \\ 0 & 1 & -1 & 0 & -4 & -3 \\ 0 & -3 & 3 & 1 & 9 & 7 \end{array}\right] \quad R_2 = -\tfrac{1}{3} r_2$$

$$\to \left[\begin{array}{rrrrr|r} 1 & 2 & 1 & 2 & -3 & -3 \\ 0 & 1 & -1 & 0 & -4 & -3 \\ 0 & 0 & 0 & 1 & -3 & -2 \end{array}\right] \quad R_3 = r_3 + 3r_2$$

$$\to \left[\begin{array}{rrrrr|r} 1 & 2 & 1 & 0 & 3 & 1 \\ 0 & 1 & -1 & 0 & -4 & -3 \\ 0 & 0 & 0 & 1 & -3 & -2 \end{array}\right] \quad R_1 = r_1 - 2r_3$$

$$\to \left[\begin{array}{rrrrr|r} 1 & 0 & 3 & 0 & 11 & 7 \\ 0 & 1 & -1 & 0 & -4 & -3 \\ 0 & 0 & 0 & 1 & -3 & -2 \end{array}\right] \quad R_1 = r_1 - 2r_2.$$

This last matrix is in reduced row-echelon form.

(b) What is the solution of this linear system?

Solution: The reduced row-echelon form of the matrix corresponds to the linear system

$$\begin{array}{ccccccccc} x_1 & & + & 3x_3 & & + & 11x_5 & = & 7 \\ & x_2 & - & x_3 & & - & 4x_5 & = & -3 \\ & & & & x_4 & - & 3x_5 & = & -2 \end{array}$$

or, in solved terms,

$$\begin{array}{rcrcrcr} x_1 & = & 7 & - & 3x_3 & - & 11x_5 \\ x_2 & = & -3 & + & x_3 & + & 4x_5 \\ x_4 & = & -2 & & & + & 3x_5. \end{array}$$

Both x_3 and x_5 are "free" variables – there is no pivot in either the "x_3" or "x_5" column of the reduced row-echelon matrix. This means that both are arbitrary, and can be chosen at will. Thus the most general solution is

$$(x_1, x_2, x_3, x_4, x_5) = (7 - 3x_3 - 11x_5, -3 + x_3 + 4x_5, x_3, -2 + 3x_5, x_5)$$

or, if you prefer,

$$\begin{array}{rcrcrcr} x_1 & = & 7 & - & 3x_3 & - & 11x_5 \\ x_2 & = & -3 & + & x_3 & + & 4x_5 \\ x_3 & = & & & x_3 & & \\ x_4 & = & -2 & & & + & 3x_5 \\ x_5 & = & & & & & x_5. \end{array}$$

(The x_3 and x_5 equations, above, are unnecessary, but one should note that both x_3 and x_5 are arbitrary.)

5 (a) Find the solution to the initial value problem

$$y' = e^x y + xy, \qquad y(0) = 3.$$

Solution: This differential equation is both first-order linear and separable. We shall, of course, solve this both ways.

As a first-order linear differential equation: We begin by writing the equation in the standard form $y' + P(x)y = G(x)$:

$$y' - (e^x + x)\, y = 0.$$

Thus $P(x) = -(e^x + x) = -e^x - x$, so $\int P(x)\ dx = -e^x - \frac{1}{2}x^2$ (we omit any constant). Thus the integrating factor is

$$h(x) = e^{\int P(x)\ dx} = \exp\left(-e^x - \frac{1}{2}x^2\right)$$

(where we've written $\exp(\ \bullet\ )$ rather than $e^{\bullet}$ just to make the function readable). We multiply our differential equation by $h(x)$ to get

$$\exp\left(-e^x - \frac{1}{2}x^2\right) y' - (e^x + x)\exp\left(-e^x - \frac{1}{2}x^2\right) y = 0 \qquad \text{or} \qquad \left[\exp\left(-e^x - \frac{1}{2}x^2\right) y\right]' = 0.$$

(The left-hand side, as usual, becomes the derivative of $h(x) \cdot y$. This is due to our choice of $h(x)$, and is actually the reason for our choice of the form of $h(x)$.) Integrating both sides, we get $\exp\left(-e^x - \frac{1}{2}x^2\right) y = K$. Multiplying both sides by $\exp\left(e^x + \frac{1}{2}x^2\right)$, we can solve for y: $y = K\exp\left(e^x + \frac{1}{2}x^2\right)$.
Finally, we find K by using the initial condition $y(0) = 3$:

$$3 = K\exp\left(e^0 + \frac{1}{2}(0)^2\right) = K\exp(1 + 0) = Ke.$$

Thus $K = \frac{3}{e} = 3e^{-1}$. Our final answer is therefore

$$y = 3e^{-1}\exp\left(e^x + \frac{1}{2}x^2\right) \qquad \text{or} \qquad y = 3\exp\left(e^x + \frac{1}{2}x^2 - 1\right).$$

As a separable differential equation: To write this as a separable differential equation, we begin by writing it as $\frac{dy}{dx} = (e^x + x)\, y$. Now multiplying by dx and dividing by y gives us $\frac{dy}{y} = (e^x + x)\ dx$. Integrating, we get

$$\int \frac{dy}{y} = \int (e^x + x)\ dx \qquad \text{or} \qquad \ln(|y|) = e^x + \frac{1}{2}x^2 + C.$$

Exponentiating, we get

$$|y| = e^{\ln(|y|)} = \exp\left(e^x + \frac{1}{2}x^2 + C\right) = \exp\left(e^x + \frac{1}{2}x^2\right) \cdot e^C \qquad \text{or} \qquad y = K\exp\left(e^x + \frac{1}{2}x^2\right)$$

where $K = \pm e^C$. Now, as before, we plug in the initial condition $y(0) = 3$ to find that $K = \frac{3}{e} = 3e^{-1}$ and

$$y = 3e^{-1}\exp\left(e^x + \frac{1}{2}x^2\right) \qquad \text{or} \qquad y = 3\exp\left(e^x + \frac{1}{2}x^2 - 1\right),$$

as before.

If instead we'd plugged in our initial condition $y(0) = 3$ earlier, when we had $\ln(|y|) = e^x + \frac{1}{2}x^2 + C$, we would get $\ln(3) = e^0 + 0 + C = 1 + C$, so $C = \ln(3) - 1$. Thus $\ln(|y|) = e^x + \frac{1}{2}x^2 + \ln(3) - 1$ and so, exponentiating, $y = \exp\left(e^x + \frac{1}{2}x^2 + \ln(3) - 1\right)$. This is the same as our previous answers, since

$$\exp\left(e^x + \frac{1}{2}x^2 + \ln(3) - 1\right) = \exp\left(e^x + \frac{1}{2}x^2 - 1\right) \cdot e^{\ln(3)} = \exp\left(e^x + \frac{1}{2}x^2 - 1\right) \cdot 3.$$

It simply looks different, but (as you can see) if we use the properties of exponentials and logarithms we may see that the answers are the same.

(b) Find the general solution to the differential equation

$$y'' - 4y' - 5y = 0.$$

Solution: To solve this second-order linear homogeneous differential equation with constant coefficients, we first find the roots of the characteristic equation $r^2 - 4r - 5 = 0$. This equation factors into $(r-5)(r+1) = 0$, so the roots are $r = 5$ and $r = -1$. Our general solution is therefore $y = Pe^{5t} + Qe^{-1t}$, where P and Q are arbitrary constants.

(c) Find the solution to the initial value problem

$$y'' - 4y' + 5y = 0, \qquad y(0) = 2, \qquad y'(0) = 1.$$

Note: This is *not* the differential equation from part (b).

Solution: To solve this second-order linear homogeneous differential equation with constant coefficients, we first find the roots of the characteristic equation $r^2 - 4r + 5 = 0$. Alas, this equation doesn't factor easily. We break out the quadratic formula and find that

$$r = \frac{+4 \pm \sqrt{(-4)^2 - 4(1)(5)}}{2(1)} = \frac{4 \pm \sqrt{-4}}{2} = \frac{4 \pm 2i}{2} = 2 \pm i.$$

We know that if the roots are complex ($r = \alpha \pm \beta i$), then the general solution is $y = e^{\alpha t}\left(P\cos(\beta t) + Q\sin(\beta t)\right)$. In this case, $\alpha = 2$ and $\beta = 1$, so our general solution is therefore $y = e^{2t}\left(P\cos(t) + Q\sin(t)\right)$.

Now we plug in our initial conditions. Since $y(0) = 2$, we get

$$2 = e^{2(0)}\left(P\cos(0) + Q\sin(0)\right) = 1 \cdot (P \cdot 1 + Q \cdot 0) = P.$$

Thus $P = 2$.

To plug in the other initial condition, we need to take the derivative first. This involves the product rule:

$$y' = 2e^{2t}\left(P\cos(t) + Q\sin(t)\right) + e^{2t}\left(-P\sin(t) + Q\cos(t)\right).$$

Now we plug in $y'(0) = 1$:

$$1 = 2e^{2(0)}\left(P\cos(0) + Q\sin(0)\right) + e^{2(0)}\left(-P\sin(0) + Q\cos(0)\right) = 2(1)\left(P \cdot 1 + Q \cdot 0\right) + (1)\left(-P \cdot 0 + Q \cdot 1\right) = 2P + Q.$$

Since $P = 2$, we get $1 = 2(2) + Q$ or $Q = -3$. Thus our final answer is

$$y = e^{2t}\big(2\cos(t) - 3\sin(t)\big).$$