Exam Two Solutions

1 Scores on the February 2006 Bar Exam had a normal distribution with mean score 137 and standard deviation 15.

(a) Find the probability that a law student scored between 125 and 152 on the exam.

Solution: This question asks for $\Pr(125 \leq X \leq 152)$, where X is a normally distributed random variable with mean $\mu = 137$ and standard deviation $\sigma = 15$. In the usual way, we convert this to a question about the standard normal distribution Z:

$$\Pr(125 \leq X \leq 152) = \Pr\left(\frac{125-137}{15} \leq Z \leq \frac{152-137}{15}\right) = \Pr\left(-0.8 \leq Z \leq 1\right).$$

The table will give us values of $\Pr(0 \leq Z \leq z)$ for positive values of z (to two digits). Thus we need to use the symmetry of the graph of the standard normal probability density function to write this probability in terms of only positive z's:

$$\begin{aligned}\Pr(125 \leq X \leq 152) &= \Pr\left(-0.8 \leq Z \leq 1\right)\\ &= \Pr\left(-0.8 \leq Z \leq 0\right) + \Pr\left(0 \leq Z \leq 1\right)\\ &= \Pr\left(0 \leq Z \leq 0.8\right) + \Pr\left(0 \leq Z \leq 1\right).\end{aligned}$$

Both these values can be found on the table: $\Pr(0 \leq Z \leq 0.8) = 0.2881$ and $\Pr(0 \leq Z \leq 1) = 0.3413$. Thus

$$\Pr(125 \leq X \leq 152) = \Pr\left(0 \leq Z \leq 0.8\right) + \Pr\left(0 \leq Z \leq 1\right) = 0.2881 + 0.3413 = 0.6294.$$

That is, the probability that a law student scored between 125 and 152 on the exam is roughly 62.94%.

(b) Find the probability that a law student scored above 125 on the exam.

Solution: This asks for the probability $\Pr(X \geq 125)$, where again X is normally distributed with mean $\mu = 137$ and standard deviation $\sigma = 15$. Again we turn this into a question about the standard normal distribution Z:

$$\Pr(X \geq 125) = \Pr\left(Z \geq \frac{125-137}{15}\right) = \Pr\left(Z \geq -0.8\right).$$

We can write this probability as $\Pr(-0.8 \leq Z \leq 0) + \Pr(Z \geq 0)$, then use the symmetry of the graph to notice two facts: first, that $\Pr(Z \geq 0) = 0.5$ (half the area of the graph is to the right of the mean), and second, that $\Pr(-0.8 \leq Z \leq 0) = \Pr(0 \leq Z \leq 0.8) = 0.2881$, as in part (a). Thus

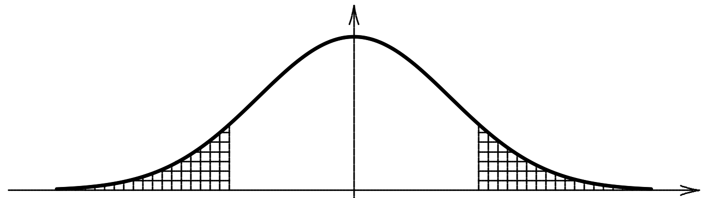
$$\Pr(X \geq 125) = \Pr\left(-0.8 \leq Z\right) = \Pr\left(0 \leq Z \leq 0.8\right) + \Pr\left(0 \leq Z\right) = 0.2881 + 0.5 = 0.7881.$$

That is, the probability that a law student scored above 125 on the exam is about 78.81%.

(c) Horace was not the best of law students. What score did he need on the bar exam to guarantee that he was NOT in the bottom 10% of all test takers in 2006? (Round this score to the nearest integer.)

Solution: Here we're looking for a score x so that $\Pr(X \leq x) = 0.10$, or 10%. As usual, we turn this into a question about Z, the standard normal distribution: $\Pr(X \leq x) = \Pr\left(Z \leq \frac{x-137}{15}\right) = 0.10$. So now we need to find $z = \frac{x-137}{15}$ with $\Pr(Z \leq z) = 0.10$.

Here's a picture of the standard normal distribution with the point z indicated:



Note that the area under the curve to the left of z is 0.10. By symmetry, the area under the curve to the right of $|z| = -z$ is also 0.10, or $\Pr(Z \geq |z|) = 0.10$. The standard normal table, however, will only show us values like $\Pr(0 \leq Z \leq |z|)$. But the area to the right of 0 is 0.5 (by symmetry, since the total area under the curve is 1), so the area between 0 and $|z|$ is 0.4: $\Pr(0 \leq Z \leq |z|) = \Pr(0 \leq Z) - \Pr(Z \geq |z|) = 0.50 - 0.10 = 0.40$. Alas, nothing on the table gives us a Z-value of precisely 0.4000. We can interpolate, however, to find such a $|z|$:

z	1.28	$\lvert z\rvert$	1.29
Z-value	0.3997	0.4000	0.4015

Often what's done in this situation is we interpolate between 1.28 and 1.29 to find $|z|$ with Z-value 0.4000:

$$\frac{\Delta z}{\Delta \text{ Value}} = \frac{|z| - 1.28}{0.4000 - 0.3997} = \frac{1.29 - 1.28}{0.4015 - 0.3997} \qquad \text{or} \qquad \frac{|z| - 1.28}{0.0003} = \frac{0.01}{0.0018}.$$

Thus $|z| = 1.28 + \frac{0.01}{0.0018}(0.0003) \approx 1.282$.

Now we turn to solving for x. Since $|z| = 1.282$, we get $z = -1.282$ and thus $\frac{x-137}{15} = -1.282$. Solving, we get $x = 137 - 1.282(15) \approx 117.77$. Thus hapless Horace needs a score of about 118 to guarantee that he is not in the bottom 10%.

Notice that the interpolation made very little difference. Solving for x, we get $x = 137 - 15|z|$ and the following possible values of x:

$\lvert z\rvert$	x
1.28	117.8
1.282	117.77
1.29	117.65

Thus the answer is always that Horace needs a score of 118 to keep out of the bottom decile.

2 An unfair *toonie* (a Canadian \$2 coin) with $\Pr(H) = .4$ and $\Pr(T) = .6$ is tossed 4 times. Let X be the random variable that records the number of heads.

(a) List the values of X together with the probability distribution.

Solution: Since X is the number of heads on 4 tosses, this random variable can clearly take on values from 0 heads to 4 heads. The probabilities are binomial probabilities; that is, this is really a Bernoulli trial with $n = 4$ trials and probability of "success" (or "heads") equal to 0.4. Thus, for example, the probability of three heads (or two successes) is

$$\Pr(X = 3) = b(4, 3; 0.4) = \binom{4}{3}(0.4)^3(0.6)^{4-3} = 4(0.4)^3(0.6)^1 = 0.1536.$$

The other computations are similar. We omit the details, but summarize the results in the following table:

k	4	3	2	1	0
$\Pr(X = k)$	0.0256	0.1536	0.3456	0.3456	0.1296

A good check is that the total probability is one:

$$0.0256 + 0.1536 + 0.3456 + 0.3456 + 0.1296 = 1.0000.$$

(b) Find the expected value of X.

Solution: The simplest way to do this is to recognize that this is a Bernoulli trial, and therefore the expected value is $E(X) = n \cdot p$. Here $n = 4$ trials with probability of success (or "heads") $p = 0.4$. Thus $E(X) = 4 \cdot 0.4 = 1.6$.

Another straightforward way to do this is to calculate the sum of outcomes (that's 0 through 4 heads) times the probability of each outcome. That is,

$$\begin{aligned} E(X) &= 4 \cdot \Pr(X = 4) + 3 \cdot \Pr(X = 3) + 2 \cdot \Pr(X = 2) + 1 \cdot \Pr(X = 1) + 0 \cdot \Pr(X = 0) \\ &= 4 \cdot 0.0256 + 3 \cdot 0.1536 + 2 \cdot 0.3456 + 1 \cdot 0.3456 + 0 \cdot 0.1296 \\ &= 1.6000, \end{aligned}$$

as before.

(c) A player's winnings for each possible outcome are shown in the table below.

Number of heads	4	3	2	1	0
Payoff	\$20	\$5	\$0	\$0	\$$N$

If a player pays \$2 to play, how much should he win (or lose) if no heads appear to make the game fair?

Solution: We will add the probabilities to the table of payoffs given to get:

Number of heads	4	3	2	1	0
Payoff	\$20	\$5	\$0	\$0	\$$N$
Probability	0.0256	0.1536	0.3456	0.3456	0.1296

The game is fair if our expected payoff is the same as the cost to play the game, namely \$2. So to find the value of N, we simply compute the expected payoff and set it equal to \$2:

$$\$2 = E(X) = \$20 \cdot 0.0256 + \$5 \cdot 0.1536 + \$0 \cdot 0.3456 + \$0 \cdot 0.3456 + \$N \cdot 0.1296.$$

If we simplify this, get $2 = 1.28 + 0.1296 \cdot N$, or $N = \frac{.72}{.1296} = \frac{50}{9} \approx 5.56$. Thus the payoff when no heads appear should be $\$N \approx \$5.56$.

Another way to approach this is to compute the expected *net* payoff. That is, we can look at the payoffs net of the \$2 it cost to play the game, and we end up with a table like this:

Number of heads	4	3	2	1	0
Payoff	\$18	\$3	$-\$2$	$-\$2$	$\$(N-2)$
Probability	0.0256	0.1536	0.3456	0.3456	0.1296

Now the game is fair if the expected net payoff is zero; that is, it's fair if (after accounting for the cost of playing) we don't expect to win or lose anything. So we compute in the same way as before:

$$\$0 = E(X) = \$18 \cdot 0.0256 + \$3 \cdot 0.1536 - \$2 \cdot 0.3456 - \$2 \cdot 0.3456 + \$(N-2) \cdot 0.1296.$$

We solve this and get $N - 2 = \frac{32}{9} \approx 3.56$, so $N \approx 5.56$, as before.

3 The lifespan of a light bulb is exponentially distributed with standard deviation $\sigma = 400$ hours.

(a) Find the expected lifespan of the light bulb.

Solution: Since the lifespan of the light bulb is exponentially distributed, we have a probability density function $f(x) = \lambda e^{-\lambda x}$ defined the domain $x \geq 0$. Recall that the mean is $\mu = \frac{1}{\lambda}$ and the variance is $\sigma^2 = \frac{1}{\lambda^2}$. Thus $\sigma = \frac{1}{\lambda}$ and, since μ is also $\frac{1}{\lambda}$, we get $\mu = \sigma = 400$ hours.

(b) Find the probability that a light bulb lasts less than 200 hours.

Solution: This is $\Pr(X \leq 200) = \int_0^{200} \lambda e^{-\lambda x}\, dx$. Thus we need to find λ. From part (a), we've seen that $\frac{1}{\lambda} = \sigma = 400$, so $\lambda = \frac{1}{400}$. Thus

$$\begin{aligned}\Pr(X \leq 200) &= \int_0^{200} \frac{1}{400} e^{-x/400}\, dx \\ &= -e^{-x/400}\Big|_0^{200} \\ &= \left(-e^{-200/400}\right) - \left(-e^{-0/400}\right) \\ &= -e^{-1/2} + 1.\end{aligned}$$

Thus the probability that a light bulb lasts less than 200 hours is $1 - e^{-1/2} \approx 0.3935$, or roughly 39.35%.

(c) Find the probability that a light bulb lasts more than 500 hours.

Solution: This asks for $\Pr(X \geq 500) = \int_{500}^{\infty} \lambda e^{-\lambda x}\, dx$. While this integral is not difficult, it is indefinite and thus requires a limit. It is simpler to simply note that $\Pr(X \geq 500) = 1 - \Pr(X \leq 500) = 1 - \int_0^{500} \lambda e^{-\lambda x}\, dx$. This integral is almost identical to the integral in part (b):

$$\begin{aligned}\Pr(X \leq 500) &= \int_0^{500} \frac{1}{400} e^{-x/400}\, dx \\ &= -e^{-x/400}\Big|_0^{500} \\ &= \left(-e^{-500/400}\right) - \left(-e^{-0/400}\right) \\ &= -e^{-5/4} + 1.\end{aligned}$$

From this we get that $\Pr(X \geq 500) = 1 - \left(1 - e^{-5/4}\right) = e^{-5/4}$. Thus the probability that a light bulb lasts more than 500 hours is $e^{-5/4} \approx 0.2865$, or roughly 28.65%.

4 Determine the value (if possible) of each integral. Use limits to justify your answer.

(a) $\displaystyle\int_2^\infty \frac{1}{(x-1)^2}\,dx$

Solution: This is an improper integral, and therefore it is really a limit:

$$\int_2^\infty \frac{1}{(x-1)^2}\,dx = \lim_{b\to\infty}\int_2^b \frac{1}{(x-1)^2}\,dx.$$

Let's first calculate the indefinite integral using the substitution $u = x-1$, so $du = dx$:

$$\int \frac{1}{(x-1)^2}\,dx = \int \frac{1}{u^2}\,du = \int u^{-2}\,du = -\frac{1}{u} + K = -\frac{1}{x-1} + K.$$

We then use this to compute our definite integral inside the limit:

$$\int_2^\infty \frac{1}{(x-1)^2}\,dx = \lim_{b\to\infty}\int_2^b \frac{1}{(x-1)^2}\,dx = \lim_{b\to\infty}\left(-\frac{1}{x-1}\right)\Bigg|_2^b$$

$$= \lim_{b\to\infty}\left[\left(-\frac{1}{b-1}\right) - \left(-\frac{1}{2-1}\right)\right] = \lim_{b\to\infty}\left(1 - \frac{1}{b-1}\right).$$

When b grows without bound, so does $b-1$, and so $\frac{1}{b-1}$ shrinks to zero. Thus the limit shown is $1-0$, so the value of our limit, and thus our integral, is 1.

(b) $\displaystyle\int_{-1}^2 \frac{8}{x^2}\,dx$

Solution: This integral is actually an improper integral, since the integrand $\frac{8}{x^2}$ is undefined at $x = 0$. Thus we must write this integral as the sum of two (improper) integrals:

$$\int_{-1}^2 \frac{8}{x^2}\,dx = \int_{-1}^0 \frac{8}{x^2}\,dx + \int_0^2 \frac{8}{x^2}\,dx.$$

Both of these new integrals are divergent, and thus our original integral is divergent as well.

To see that both of the integrals above are divergent, we try to compute one of them (the other is extremely similar). The first integral on the right-hand side is

$$\int_{-1}^0 \frac{8}{x^2}\,dx = \lim_{b\to 0^-}\int_{-1}^b 8x^{-2}\,dx = \lim_{b\to 0^-}\left(-8x^{-1}\Big|_{-1}^b\right) = \lim_{b\to 0^-}\left(-\frac{8}{b} + \frac{8}{-1}\right).$$

As b approaches 0 (from the left), the term $-\frac{8}{b}$ grows without bound, and so has infinite limit. Thus the limit, and therefore the improper integral, is divergent.

5 (a) Find the constant k so that $f(x) = k\sqrt{5-x}$ is a probability density function on the interval $[1,4]$.

Solution: There are two requirements for a probability density function $f(x)$: it must never be negative, and the total area under the curve (that is, the integral over its domain) must be 1. (These requirements are, roughly, that there can be no negative probability and that the total probability should be 1, or 100%.) The effect here is that k shouldn't be negative and that $\int_1^4 k\sqrt{5-x}\,dx = 1$.

We compute this integral, using the substitution $u = 5-x$, so $du = -dx$, $u = 4$ when $x = 1$, and $u = 1$ when $x = 4$:

$$1 = \int_1^4 k\sqrt{5-x}\,dx = k\int_{u=4}^{u=1} \sqrt{u}\cdot -du = k\int_{u=1}^{u=4} u^{1/2}\,du$$

$$= k\cdot \frac{2}{3}u^{3/2}\Big|_{u=1}^{u=4} = \frac{2k}{3}\left(4^{3/2} - 1^{3/2}\right) = \frac{2k}{3}\cdot 7.$$

Thus $1 = \frac{14k}{3}$, or $k = \frac{3}{14} \approx 0.2143$.

(Notice that our answer is *not* negative, so the answer we've found from the second requirement on k also satisfies the first.)

(b) Now consider the probability density function $f(x) = \dfrac{30-4x}{100}$ on the interval $[0,5]$. Find the constant C so that $\Pr(X \le C) = \dfrac{1}{2}$.

Solution: This question asks us to find the median of our probability distribution. This isn't as simple as it sounds, but it isn't too difficult. We're asked to find C so that

$$\frac{1}{2} = \Pr(X \le C) = \int_0^C \frac{30-4x}{100}\,dx.$$

Once we compute the integral on the right, our answer will still involve C, which we can then find since the integral equals one-half. We compute:

$$\frac{1}{2} = \int_0^C \frac{30-4x}{100}\,dx = \frac{1}{100}\int_0^C (30-4x)\ dx = \frac{1}{100}\left(30x - 2x^2\right)\Big|_0^C$$

$$= \frac{1}{100}\left[\left(30C - 2C^2\right) - \left(30\cdot 0 - 2(0)^2\right)\right] = \frac{1}{100}\left(30C - 2C^2\right).$$

If we multiply both sides of the resulting equation by 100 and move everything to one side, we get the equation $2C^2 - 30C + 50 = 0$, or $C^2 - 15C + 25 = 0$. We break out the quadratic formula to see that

$$C = \frac{15 \pm \sqrt{(-15)^2 - 4(1)(25)}}{2(1)} = \frac{15 \pm \sqrt{125}}{2}.$$

While it appears we have two answers, we should note that $\frac{15+\sqrt{125}}{2} \approx 13.0902$ is not in the interval $[1,4]$ on which our density function $f(x)$ is defined. Thus it can't possibly the median. Thus our answer is $C = \frac{15-\sqrt{125}}{2} \approx 1.9098$.