1 For each of the following questions, show appropriate work to support your answers.

(a) Find $\displaystyle\int \frac{3x^2+6}{2x^3+12x-1}\,dx$.

Solution: For this problem we make the substitution $u = 2x^3+12x-1$, aiming to turn the integral into a multiple of $\int \frac{1}{u}\,du$. Thus $du = \frac{du}{dx}\cdot dx = \left(6x^2+12\right)dx$, and the key is that $6x^2+12 = 2(3x^2+6)$, which is twice the term in the numerator. That is, we can write the integral as

$$\int \frac{3x^2+6}{2x^3+12x-1}\,dx = \int \frac{3x^2+6}{u}\cdot\frac{du}{2(3x^2+6)} = \int \frac{1}{u}\cdot\frac{du}{2} = \frac{1}{2}\int\frac{du}{u}.$$

This integrates to $\frac{1}{2}\ln(|u|)+K$. Writing this again in terms of x, we have

$$\int \frac{3x^2+6}{2x^3+12x-1}\,dx = \frac{1}{2}\ln\left(\left|2x^3+12x-1\right|\right)+K.$$

(b) Find $\displaystyle\int_0^3 5xe^x\,dx$.

Solution: This integral requires integration by parts. We choose $u=5x$ and $dv = e^x\,dx$, so $du = 5\,dx$ and $v = e^x$. Thus we have (maintaining the limits of the definite integral as we go)

$$\begin{aligned}\int_0^3 5xe^x\,dx &= u\cdot v\Big|_0^3 - \int_0^3 v\,du\\ &= 5xe^x\Big|_0^3 - \int_0^3 e^x\cdot 5\,dx\\ &= \left(5\cdot 3e^3 - 5\cdot 0e^0\right) - 5e^x\Big|_0^3\\ &= 15e^3 - 0 - 5\left(e^3-e^0\right) = 10e^3+5\end{aligned}$$

Thus $\displaystyle\int_0^3 5xe^x\,dx = 10e^3+5 \approx 205.8554$.

(c) Find $\displaystyle\int f(x)\,dx$, where $f'(x) = \sqrt{1+x^2}$. Your answer should involve $f(x)$.

Hint: Use integration by parts.

Solution: The hint is to use integration by parts. Since we don't really know how to integrate $f(x)$ directly (we don't even have an algebraic expression for $f(x)$), the natural choice is $u = f(x)$ and $dv = dx$. Then $du = f'(x)\,dx = \sqrt{1+x^2}\,dx$ and $v = x$, so

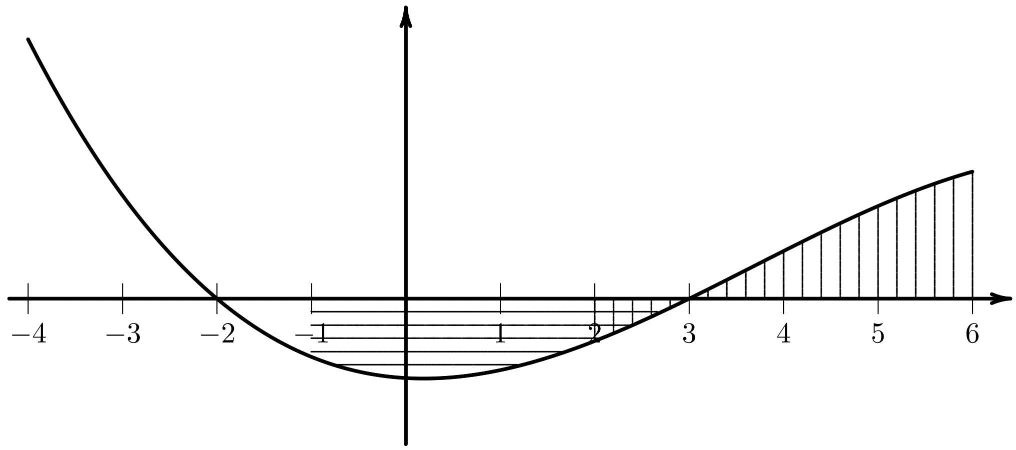
$$\int f(x)\,dx = u\cdot v - \int v\,du = f(x)\cdot x - \int x\cdot\sqrt{1+x^2}\,dx.$$

We can compute this new integral using the simple substitution $u = 1+x^2$, so $du = 2x\,dx$ and $dx = \frac{du}{2x}$ and

$$\int x\sqrt{1+x^2}\,dx = \int x\sqrt{u}\cdot\frac{du}{2x} = \frac{1}{2}\int u^{1/2}\,du = \frac{1}{2}\cdot\frac{2}{3}u^{3/2}+K = \frac{1}{3}(1+x^2)^{3/2}+K.$$

Thus $\displaystyle\int f(x)\,dx = xf(x) - \frac{1}{3}(1+x^2)^{3/2}+C$.

2 Let $f(x)$ be the function shown in the graph below. In each case, determine if the given definite integral is positive, negative, or zero. Explain your reasoning.



(a) $\displaystyle\int_{-1}^{3} f(x)\,dx$

Solution: This integral is negative. This definite integral represents the horizontally shaded area, above. This area is entirely below the x-axis, so it has area that is counted negatively by the integral. Since the integral is only seeing "negative" area, the definite integral is negative.

(b) $\displaystyle\int_{2}^{6} f(x)\,dx$

Solution: This integral is positive. This definite integral represents the vertically shaded area, above. This area is partly below the x-axis (from $x = 2$ to $x = 3$) but mostly above the x-axis (from $x = 3$ to $x = 6$). Thus a small part of the area is counted negatively, but the larger part of the area is counted positively by the integral. Thus the definite integral is positive.

(c) $\displaystyle\int_{2}^{2} f(x)\,dx$

Solution: This integral is zero. This is simply a property of integrals: if the upper and lower limits are the same, the integral is zero. In terms of area, this is the area under $x = 2$. But this area has no width, so it has zero area.

Put another way, if $F(x)$ is an anti-derivative of $f(x)$, then the given definite integral is $F(b) - F(a) = F(2) - F(2) = 0$.

3 Suppose that a continuous function $f(x)$ has average value equal to 3 over the interval $[-4, 7]$. Find $\displaystyle\int_{-4}^{7} f(x)\,dx$.

Solution: Recall that the average value of a continuous function $f(x)$ over an interval $a \le x \le b$ is

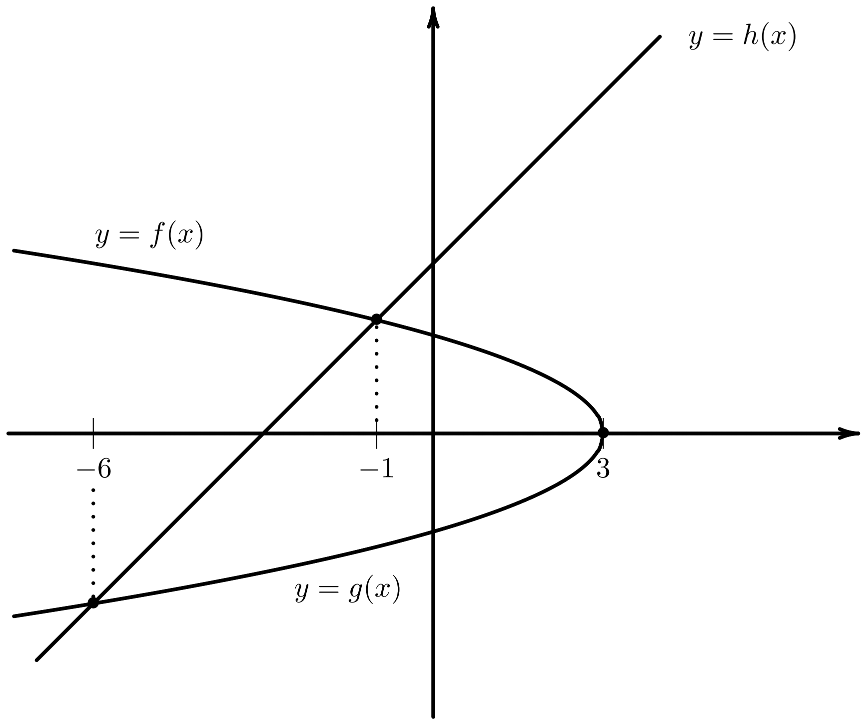
$$\text{Ave.} = \frac{1}{b-a}\int_a^b f(x)\,dx.$$

Here we're given that the average is 3 and that $b - a = 7 - (-4) = 11$, so the above formula can be re-written as

$$3 = \frac{1}{11}\int_{-4}^{7} f(x)\,dx \qquad \text{or} \qquad \int_{-4}^{7} f(x)\,dx = 3 \cdot 11 = 33.$$

Thus $\displaystyle\int_{-4}^{7} f(x)\,dx = 33$.

4 Write down an integral (or integrals) that represents the area of the region bounded by the graphs of $f(x) = \sqrt{3-x}$, $g(x) = -\sqrt{3-x}$, and $h(x) = x + 3$. You do not need to evaluate the integral(s)! For your convenience, the graphs of these functions are shown below:



Solution: To find the required area, we need to find where the curves cross. These points of intersection are found by setting the equations of the curves equal to each other:

$$x + 3 = \pm\sqrt{3-x}.$$

We solve for these points by squaring both sides, then putting everything on one side:

$$(x+3)^2 = \left(\pm\sqrt{3-x}\right)^2 \qquad \text{or} \qquad x^2 + 6x + 9 = 3 - x$$

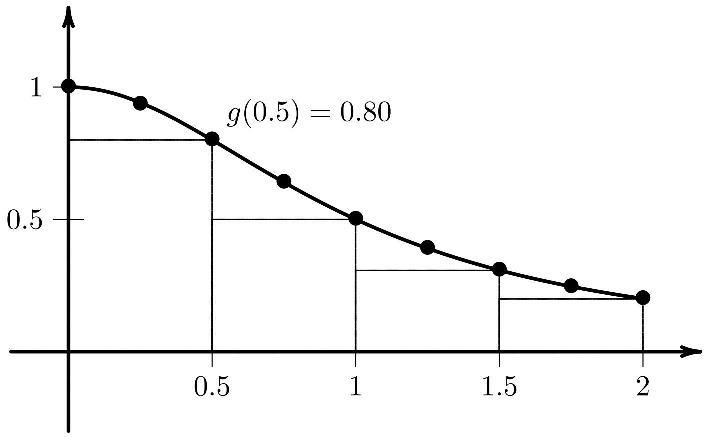
This simplifies to $x^2 + 7x + 6 = 0$, which factors into $(x+6)(x+1) = 0$. Thus the curves intersect at $x = -6$ (the point $(x,y) = (-6,-3)$ is where $h(x) = x+3$ and $g(x) = -\sqrt{3-x}$ cross) and $x = -1$ (the point $(x,y) = (-1,2)$ is where $h(x) = x+3$ and $f(x) = \sqrt{3-x}$ meet). The two curves $f(x) = \sqrt{3-x}$ and $g(x) = -\sqrt{3-x}$ also meet at $(x,y) = (3,0)$.

The area bounded by these three curves therefore has $g(x) = -\sqrt{3-x}$ as the bottom curve for the entire region. The top curve is $h(x) = x+3$ from $x = -6$ to $x = -1$, and $f(x) = \sqrt{3-x}$ from $x = -1$ to $x = -6$. Thus the total area requested is

$$\begin{aligned}
\text{Area} &= \int_{-6}^{-1} \left(h(x) - g(x)\right)\, dx + \int_{-1}^{3} \left(f(x) - g(x)\right)\, dx \\
&= \int_{-6}^{-1} \left[(x+3) - \left(-\sqrt{3-x}\right)\right]\, dx + \int_{-1}^{3} \left[\left(\sqrt{3-x}\right) - \left(-\sqrt{3-x}\right)\right]\, dx \\
&= \int_{-6}^{-1} \left(x + 3 + \sqrt{3-x}\right)\, dx + \int_{-1}^{3} 2\sqrt{3-x}\, dx.
\end{aligned}$$

We were not asked to find this area, but if you did you'd find it was $\frac{125}{6} \approx 20.8333$.

5 Let $g(x)$ be the graph shown below. Some specific values of $g(x)$ are shown in the tables to the right.



x	$g(x)$
0	1
0.25	0.95
0.5	0.80
0.75	0.65
1	0.50

x	$g(x)$
1.25	0.40
1.5	0.30
1.75	0.25
2	0.20

(a) Approximate the definite integral $\int_0^2 g(x)\ dx$ by partitioning the interval $[0, 2]$ into four subintervals of equal length and choosing u as the right endpoint of each subinterval.

Solution: The requested approximation is the one that breaks up the interval $[0, 2]$ into $n = 4$ rectangles, each of width $\Delta x = \frac{b-a}{n} = \frac{2-0}{4} = 0.5$. These four rectangles are drawn on the graph, above, and they have width $\Delta x = 0.5$ and height $g(0.5)$ through $g(2)$ (see the label for $g(0.5)$ on the curve). Thus we approximate the integral (the area under the curve) with the area of these four rectangles:

$$\begin{aligned} \text{Area} &= \Delta x \cdot g(0.5) + \Delta x \cdot g(1) + \Delta x \cdot g(1.5) + \Delta x \cdot g(2) \\ &= \Delta x \cdot (g(0.5) + g(1) + g(1.5) + g(2)) \\ &= 0.5\,(0.80 + 0.50 + 0.30 + 0.20) \\ &= 0.90. \end{aligned}$$

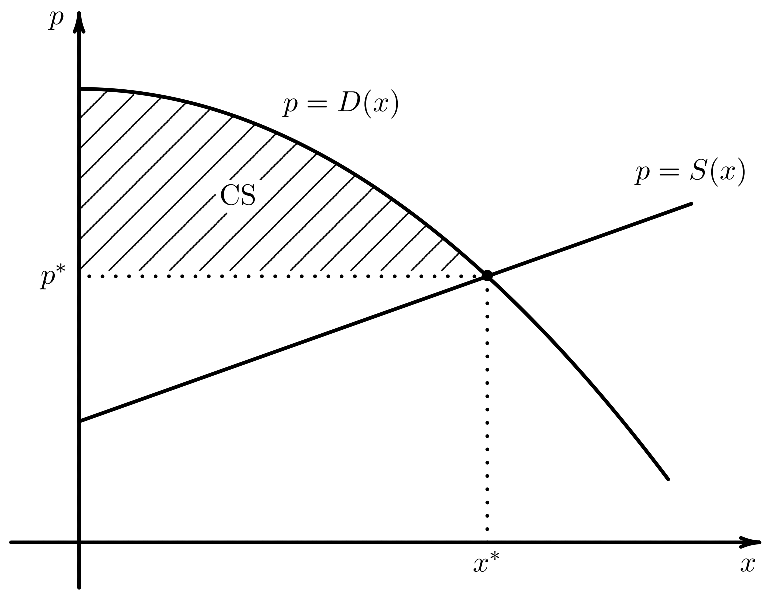
This is our approximation: $\int_0^2 g(x)\ dx \approx 0.90$.

(b) Will your estimate be greater than, less than, or equal to the exact area under the curve? Explain how you know this without knowing the actual value of the integral.

Solution: The estimate is less than the actual area, as is made clear by the boxes drawn above. The estimate is the area of the boxes which excludes some of the area underneath the curve. This happens because the function is decreasing on the interval $[0, 2]$, so the right endpoint of each subinterval is the lowest the curve gets on that subinterval.

6 The graph below shows the demand curve $p = D(x)$ and the supply curve $p = S(x)$ for I♡MATH t-shirts. Clearly shade in the region whose area is the consumer's surplus (CS).

Solution:



7 In 2004 the population of Ljubljana (the capital of Slovenia) was 265, 000. Census data predicts the population will continue to grow at a rate of $100 + t^{2/3}$ people per year, where t represents the number of years elapsed since 2004. Predict the size of Ljubljana's population in 2010.

Solution: If we let $P(t)$ be the population of Ljubljana t years after 2004, then we're told two things: $P'(t) = 100 + t^{2/3}$ people per year and $P(0) = 265,000$. We're asked for $P(6)$, since 2010 is $t = 6$ years after 2004.

We find $P(t)$ by integrating $P'(t)$. That is,

$$P(t) = \int P'(t)\, dt = \int \left(100 + t^{2/3}\right)\, dt = 100t + \frac{1}{5/3}t^{5/3} + K = 100t + \frac{3}{5}t^{5/3} + K.$$

We find K by plugging in the initial condition: $P(0) = 100\cdot 0 + \frac{3}{5}\cdot 0^{5/3} + K = 265,000$, so $K = 265,000$. Thus the population model is $P(t) = 100t + \frac{3}{5}t^{5/3} + 265,000$, so in 2010 the population of Ljubljana will be

$$P(6) = 100(6) + \frac{3}{5}(6)^{5/3} + 265,000 \approx 265,611.8869,$$

or about 265, 612 people.