Solving Systems Using Inverses

For each of the following problems...

(a) Write the system as $AX = B$, where $X = \begin{bmatrix} x \\ y \end{bmatrix}$ or $X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$.

(b) Find the inverse matrix A^{-1} if it exists.

(c) If you were able to find A^{-1} in part (b), then solve the system by writing $X = A^{-1}B$.

1
$$\begin{aligned} 2x - 3y &= -1 \\ 4x + 6y &= 6 \end{aligned}$$

2
$$\begin{aligned} 3x - y &= 8 \\ 2x + 3y &= 9 \end{aligned}$$

3
$$\begin{aligned} 2x - y &= 6 \\ 4x - 2y &= 12 \end{aligned}$$

4
$$\begin{aligned} 2x - y &= 6 \\ 4x - 2y &= 2 \end{aligned}$$

5
$$\begin{aligned} x + 2y &= 11 \\ 4x - y &= -1 \end{aligned}$$

6
$$\begin{aligned} x - 2y + 4z &= 6 \\ -2x + 3y - 6z &= -14 \\ 3x + y - z &= -1 \end{aligned}$$

7
$$\begin{aligned} x + 2y + z &= 2 \\ 2x + y + 3z &= 1 \\ -2x + 3y - 4z &= 1 \end{aligned}$$

8
$$\begin{aligned} x + y + z &= 4 \\ x - y - 2z &= 6 \\ x + 5y + 7z &= 12 \end{aligned}$$

9
$$\begin{aligned} x + y + z &= 4 \\ x - y - 2z &= 6 \\ x + 5y + 9z &= 12 \end{aligned}$$

Solving Systems – Answers

1 $A^{-1} = \frac{1}{24}\begin{bmatrix} 6 & 3 \\ -4 & 2 \end{bmatrix}$ & $X = \begin{bmatrix} x \\ y \end{bmatrix} = A^{-1}B = \frac{1}{24}\begin{bmatrix} 6 & 3 \\ -4 & 2 \end{bmatrix}\begin{bmatrix} -1 \\ 6 \end{bmatrix} = \frac{1}{24}\begin{bmatrix} 12 \\ 16 \end{bmatrix} = \begin{bmatrix} 1/2 \\ 2/3 \end{bmatrix}$

2 $A^{-1} = \frac{1}{11}\begin{bmatrix} 3 & 1 \\ -2 & 3 \end{bmatrix}$ & $X = \begin{bmatrix} x \\ y \end{bmatrix} = A^{-1}B = \frac{1}{11}\begin{bmatrix} 3 & 1 \\ -2 & 3 \end{bmatrix}\begin{bmatrix} 8 \\ 9 \end{bmatrix} = \frac{1}{11}\begin{bmatrix} 33 \\ 11 \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$

3 There are infinitely many solutions. In particular, there is no inverse matrix A^{-1} since $ad-bc = (2)(-2) - (-1)(4) = 0$.

4 The system is inconsistent. Again there is no inverse matrix, since $ad - bc = (2)(-2) - (-1)(4) = 0$. (It's the same matrix A as before. This demonstrates that $AX = B$ and $AX = C$ can have either zero solutions or infinitely many *for the same matrix A!*)

5 $A^{-1} = \frac{1}{9}\begin{bmatrix} 1 & 2 \\ 4 & -1 \end{bmatrix}$ & $X = \begin{bmatrix} x \\ y \end{bmatrix} = A^{-1}B = \frac{1}{9}\begin{bmatrix} 1 & 2 \\ 4 & -1 \end{bmatrix}\begin{bmatrix} 11 \\ -1 \end{bmatrix} = \frac{1}{9}\begin{bmatrix} 9 \\ 45 \end{bmatrix} = \begin{bmatrix} 1 \\ 5 \end{bmatrix}$

6 $A^{-1} = \begin{bmatrix} -3 & -2 & 0 \\ 20 & 13 & 2 \\ 11 & 7 & 1 \end{bmatrix}$ & $X = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = A^{-1}B = \begin{bmatrix} -3 & -2 & 0 \\ 20 & 13 & 2 \\ 11 & 7 & 1 \end{bmatrix}\begin{bmatrix} 6 \\ -14 \\ -1 \end{bmatrix} = \begin{bmatrix} 10 \\ -64 \\ -33 \end{bmatrix}$

7 $A^{-1} = \begin{bmatrix} 13 & -11 & -5 \\ -2 & 2 & 1 \\ -8 & 7 & 3 \end{bmatrix}$ & $X = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 13 & -11 & -5 \\ -2 & 2 & 1 \\ -8 & 7 & 3 \end{bmatrix}\begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 10 \\ -1 \\ -6 \end{bmatrix}$

8 The system is inconsistent and A has no inverse. If we were to try to solve this using row reduction, we'd get

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 4 \\ 1 & -1 & -2 & 6 \\ 1 & 5 & 7 & 12 \end{array}\right] \longrightarrow \left[\begin{array}{ccc|c} 1 & 1 & 1 & 4 \\ 0 & -2 & -3 & 2 \\ 0 & 4 & 6 & 8 \end{array}\right] \quad \begin{array}{l} R_2 = r_2 - r_1 \\ R_3 = r_3 - r_1 \end{array}$$

$$\longrightarrow \left[\begin{array}{ccc|c} 1 & 1 & 1 & 4 \\ 0 & 1 & 3/2 & -1 \\ 0 & 4 & 6 & 8 \end{array}\right] \quad R_2 = -\tfrac{1}{2} \cdot r_2$$

$$\longrightarrow \left[\begin{array}{ccc|c} 1 & 1 & 1 & 4 \\ 0 & 1 & 3/2 & -1 \\ 0 & 0 & 0 & 12 \end{array}\right] \quad R_3 = r_3 - 4r_2$$

9 $A^{-1} = \frac{1}{4}\begin{bmatrix} -1 & 4 & 1 \\ 11 & -8 & -3 \\ -6 & 4 & 2 \end{bmatrix}$ & $X = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \frac{1}{4}\begin{bmatrix} -1 & 4 & 1 \\ 11 & -8 & -3 \\ -6 & 4 & 2 \end{bmatrix}\begin{bmatrix} 4 \\ 6 \\ 12 \end{bmatrix} = \frac{1}{4}\begin{bmatrix} 32 \\ -40 \\ 24 \end{bmatrix} = \begin{bmatrix} 8 \\ -10 \\ 6 \end{bmatrix}$