# Homework: Trigonometric Functions



## Answer Key

**1** $-\frac{1}{3}\cos(x^3+1)+K$

**2** $-\sin(1/x)+K$

**3** $\ln(|\sin(t)|)+K$

**4** $\frac{1}{3}\sin^3(x)+K$

**5** $\frac{1}{3}\sin^3(x)-\frac{1}{5}\sin^5(x)+K$

**6** $\frac{1}{3}x^3\sin(x^3)+\frac{1}{3}\cos(x^3)+K$

**7** $\frac{3}{20}\left(10+5\sin(2x)\right)^{2/3}+K$

**8** $-\frac{1}{3}\cos(e^{3x})+K$

**9** $\frac{1}{4}e^{\sin(4x)}+K$

**10** Answers will vary.

**11** Answers will vary, but since this is optional I'll give many details.

(a) We integrate by parts. We'll choose $u=e^{ax}$ and so $dv=\cos(bx)\,dx$. Thus

$$u=e^{ax} \qquad dv=\cos(bx)\,dx$$
$$du=ae^{ax}\,dx \qquad v=\int\cos(bx)\,dx=\frac{1}{b}\sin(bx).$$

Now we perform the integration by parts:

$$\begin{aligned}\int e^{ax}\cos(bx)\,dx &= e^{ax}\cdot\frac{1}{b}\sin(bx)-\int\frac{1}{b}\sin(bx)\cdot ae^{ax}\,dx\\ &=\frac{1}{b}e^{ax}\sin(bx)-\frac{a}{b}\int e^{ax}\sin(bx)\,dx. \qquad (*)\end{aligned}$$

Now we integrate this new integral by parts. The key to making this whole process work is that we use the same $u$:

$$u=e^{ax} \qquad dv=\sin(bx)\,dx$$
$$du=ae^{ax}\,dx \qquad v=\int\sin(bx)\,dx=-\frac{1}{b}\cos(bx).$$

(Notice that while $u$ is unchanged, $dv$ has necessarily changed since we have a different integral.) Now we integrate the new integral from equation $(*)$:

$$\begin{aligned}\int e^{ax}\sin(bx)\,dx &= e^{ax}\cdot\left(-\frac{1}{b}\cos(bx)\right)-\int\left(-\frac{1}{b}\cos(bx)\right)\cdot ae^{ax}\,dx\\ &=-\frac{1}{b}e^{ax}\cos(bx)+\frac{a}{b}\int e^{ax}\cos(bx)\,dx. \qquad (**)\end{aligned}$$

Putting together equations $(*)$ and $(**)$, we get

$$\begin{aligned}\int e^{ax}\cos(bx)\,dx &= \frac{1}{b}e^{ax}\sin(bx) - \frac{a}{b}\left[-\frac{1}{b}e^{ax}\cos(bx) + \frac{a}{b}\int e^{ax}\cos(bx)\,dx\right] \\ &= \frac{1}{b}e^{ax}\sin(bx) + \frac{a}{b^2}e^{ax}\cos(bx) - \frac{a^2}{b^2}\int e^{ax}\cos(bx)\,dx.\end{aligned}$$

Now we add $\frac{a^2}{b^2}\int e^{ax}\cos(bx)\,dx$ to both sides of the equation to get

$$\int e^{ax}\cos(bx)\,dx + \frac{a^2}{b^2}\int e^{ax}\cos(bx)\,dx = \frac{1}{b}e^{ax}\sin(bx) + \frac{a}{b^2}e^{ax}\cos(bx)$$

or

$$\begin{aligned}\left(1+\frac{a^2}{b^2}\right)\int e^{ax}\cos(bx)\,dx &= \frac{1}{b}e^{ax}\sin(bx) + \frac{a}{b^2}e^{ax}\cos(bx) \\ \frac{a^2+b^2}{b^2}\int e^{ax}\cos(bx)\,dx &= \frac{1}{b}e^{ax}\sin(bx) + \frac{a}{b^2}e^{ax}\cos(bx).\end{aligned}$$

After multiplying through by $\frac{b^2}{a^2+b^2}$, we get

$$\begin{aligned}\int e^{ax}\cos(bx)\,dx &= \frac{b^2}{a^2+b^2}\cdot\frac{1}{b}e^{ax}\sin(bx) + \frac{b^2}{a^2+b^2}\cdot\frac{a}{b^2}e^{ax}\cos(bx) \\ &= \frac{b}{a^2+b^2}e^{ax}\sin(bx) + \frac{a}{a^2+b^2}e^{ax}\cos(bx) \\ &= \frac{1}{a^2+b^2}e^{ax}\left(b\sin(bx) + a\cos(bx)\right).\end{aligned}$$

This is the formula we wanted (provided we add on the arbitrary constant $K$).

(b) For this part, equation $(**)$ is the integration by parts we are supposed to do:

$$\int e^{ax}\sin(bx)\,dx = -\frac{1}{b}e^{ax}\cos(bx) + \frac{a}{b}\int e^{ax}\cos(bx)\,dx.$$

Now we can plug in the formula for $\int e^{ax}\cos(bx)\,dx$ that we derived in part (a):

$$\begin{aligned}\int e^{ax}\sin(bx)\,dx &= -\frac{1}{b}e^{ax}\cos(bx) + \frac{a}{b}\int e^{ax}\cos(bx)\,dx \\ &= -\frac{1}{b}e^{ax}\cos(bx) + \frac{a}{b}\left[\frac{1}{a^2+b^2}e^{ax}\left(b\sin(bx) + a\cos(bx)\right)\right] \\ &= \left(\frac{a^2}{b(a^2+b^2)} - \frac{1}{b}\right)e^{ax}\cos(bx) + \frac{a}{a^2+b^2}e^{ax}\sin(bx) \\ &= \left(\frac{a^2}{b(a^2+b^2)} - \frac{a^2+b^2}{b(a^2+b^2)}\right)e^{ax}\cos(bx) + \frac{a}{a^2+b^2}e^{ax}\sin(bx) \\ &= \frac{-b}{a^2+b^2}e^{ax}\cos(bx) + \frac{a}{a^2+b^2}e^{ax}\sin(bx) \\ &= \frac{1}{a^2+b^2}e^{ax}\left(-b\cos(bx) + a\sin(bx)\right).\end{aligned}$$

This is the formula we wanted (provided we again add on the arbitrary constant $K$).

12 (a) $\frac{1}{25}e^{3x}\big(3\cos(4x) + 4\sin(4x)\big) + K$

(b) $-\frac{1}{2}e^{-x}\big(\sin(x) + \cos(x)\big) + K$

(c) $\frac{1}{50}e^{7x}\big(7\cos(x) + \sin(x)\big) + K$