Math 126 **Final Exam Review** Spring, 2008

These problems are (mostly) selections from old final exams and are of the type you can expect to find on the final exam (in Ford Auditorium on Friday, May 2nd, from 4:00 PM to 7:00 PM). Solutions will be posted on the class web page: `http://filer.case.edu/pmg5/126/`.

1 Evaluate the following integrals:

(a) $\int \frac{x}{2} \sin(x/2)\ dx$ (b) $\int_{-1}^{1} x^2\sqrt{1-x^3}\ dx$

2 The standard normal curve may be written $f(z) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}z^2}$. The area under $f(z)$ from 0 to z may be written $A(z) = \int_0^z f(z)dz$.

For a random variable Z having a standard normal density, the probability $\Pr(-1 \le Z \le 1) \approx 0.68$ (this is the probability that Z is within one standard deviation of the mean). This value is the area under $f(z)$ from -1 to 1.

Estimate this area (or probability) in two ways:

(a) Divide the interval $[-1, 1]$ into $n = 4$ subintervals and use the right-hand endpoint of each subinterval as u.

(b) Notice that $f(z)$ is symmetric (that is, $f(z) = f(-z)$; draw the picture for yourself), so the area from $z = -1$ to $z = 0$ is the same as the area from $z = 0$ to $z = 1$. So estimate the area from 0 to 1 (using $n = 4$ subintervals and the right-hand endpoint, as in part (a)), then double that value.

3 Solve the following initial value problem: $y' = -y\ \sin(\pi t)$, $y(\frac{1}{2}) = -\frac{1}{2}$

4 (a) For what value k is $f(x) = k\cos\left(\frac{\pi}{2}x\right)$ a probability function on the interval $-1 \le x \le 1$?

(b) If X is a continuous random variable whose probabilities follow this density and distribution, find $\Pr(X > \frac{1}{2})$.

5 An experiment is performed to study the evaporation of alcohol. The following data are collected:

Time after start of experiment (x or t in hours)	0	2	4	6	8	10
Amount of alcohol remaining (y in liters)	1.00	0.85	0.72	0.59	0.45	0.31

(a) Use the method of linear regression (least squares) to find a formula $y = mx + b$ that describes the process over time. (Keep 4 places past the decimal in your calculations.)

(b) Use the model equation you have derived in (a) to predict the time when there is no alcohol remaining.

6 Automobile tires are considered unsafe once the depth of the tread decreases below a certain point. Consider an exponential random variable X that is the safe lifetime (in thousands of miles driven) of a randomly selected tire. Bargain Brand Tires have a particular basic model whose average safe lifetime is 10 thousand miles. (The fact that a car has 4 tires need not be taken into account.)

(a) What is the value of the parameter k for this exponential density?

(b) Find the probability that a randomly selected tire lasts more than 10,000 miles?

(c) You are going on 10,000 mile adventure around North and Central America. Assume that driving on an unsafe tire is certain to cause a flat. Further assume that having a flat repaired on this particular trip will end up costing \$100.

Your local tire dealer sells Bargain Brand basic model tires for \$75 each (you only need one for this problem). They also offer a \$40 protection plan, so that if you have a flat anywhere on your trip all tire related repairs and towing are covered. Finally, the salesman shows you a deluxe Bargain Brand model for \$110 with an average lifetime of 25 thousand miles and the protection plan thrown in free. **Should you a) buy the basic model and the protection plan, b) buy the basic model without the protection plan or c) buy the deluxe model?** Back up your answer with the appropriate calculation.

7 Let $A = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 1 \\ 0 & 1 & 2 \end{bmatrix}$.

(a) Find A^{-1} by hand (show your calculations).

(b) Use part (a) to solve the following system:

$$\begin{aligned} x \quad\;\; + \;\; z &= \;\;1 \\ x + y + \;\; z &= -1 \\ y + 2z &= \;\;0 \end{aligned}$$

(c) If one changes the number 2 in matrix A to some other number, say k, one can make the new matrix have no inverse. Find k.

8 Let $A = \begin{bmatrix} 1 & 3 \\ 2 & 6 \end{bmatrix}$ and $B = \begin{bmatrix} 5 & 3 \\ 4 & 6 \end{bmatrix}$.

(a) Does $(A+B)^{-1}$ exist? Verify your answer with an appropriate calculation.

(b) Does $(A-B)^{-1}$ exist? Verify your answer with an appropriate calculation.

(c) The problem $AX - BX = Y$ is equivalent to $(A-B)X = Y$. Let $Y = \begin{bmatrix} -10 \\ -5 \end{bmatrix}$.

Say elimination (row reduction) is applied to this problem. The augmented matrix $[(A-B) \mid Y]$ is reduced by forward elimination to the matrix $\left[\begin{array}{cc|c} 1 & 0 & 5/2 \\ 0 & 0 & 0 \end{array}\right]$.

Write an expression for the solution(s) of this system. If no solutions exist, then state that along with something to back it up.

(d) Given A and B above and your previous results, can you say if the system in (c) has a unique solution? Explain.

9 Let

$$A = \begin{bmatrix} 1 & 1 & -1 \\ 2 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix}.$$

Does A^{-1} exist? Explain how you know. If it exists, find A^{-1}.

10 Data giving the number of hours a student has studied compared to his or her performance on an exam (of maximum 100 points) are given in the following table:

Hours Studied	0	3	6	9
Exam Score	50	62	81	92

(a) Find a least squares line that best fits the given data.

(b) What prediction would this line make for a student who studied 10 hours?

11 Suppose an assembly line produces a small mechanical toy, and about 1% of the toys are defective. A quality control worker chooses a toy at random, inspects it, and then repeats this process until a defective toy is found.

(a) Find the probability that at most 2 toys pass the test before a defective toy is found.

(b) What is the probability that at least 3 toys will pass the test before the first defective toy is found?

12 The SACT is a new test for college-bound high school students. It is designed so that scores are normally distributed with a mean of 400 and a standard deviation of 100.

(a) Find the probability that a randomly selected high school student scores between 300 and 450 on the SACT.

(b) Find the probability that a randomly selected high school student scores over 650 on the SACT.

(c) It turns out that there was a mistake in the computations. The mean of the SACT is 400, but the standard deviation is *not* 100. One school is told that a 520 is the cut-off for scores in the top 5%. Assuming this is correct, what is the standard deviation?

13 (a) Find the area of the region bounded by the curves $y = x^2 - x$ and $y = x + 3$.

(b) Evaluate the integral $\displaystyle\int t\cos(2t)\,dt$.

(c) Evaluate the integral $\displaystyle\int_0^1 x\sqrt{1-x}\,dx$.

14 We are given the differential equation $y'' - 6y' + cy = 0$, where c is an unknown constant. Find c so that...

(a) $\ldots y = 6te^{3t}$ is a solution.

(b) $\ldots y = 2e^{t} + 4e^{5t}$ is a solution.

(c) $\ldots y = 2e^{3t}\cos(2t)$ is a solution.

15 The telephone company performed a survey over several years to find what percentage of the population still use rotary (dial) telephones. They collected the following data:

Year of survey (x or t)	1	2	3	4
Percent using rotary phones (y)	20	15	5	4

(a) Use the method of linear regression (least squares) to find a formula $y = mx + b$ that describes the percentage using rotary phones in year x. (Keep 4 digits past the decimal in your calculations.)

(b) Use the model equation you have derived in (a) to predict the year when the percentage using rotary phones is 0.

16 Suppose the average life span of a new cellular phone is 50 months, and that these life spans are exponentially distributed.

(a) Find the probability density function $f(x)$ for this distribution.

(b) Find the probability that a cellular phone will last more than 2 years (24 months).

(c) Find the probability that a cellular phone will fail within the first 2 years (24 months).

17 (a) For what value of k will $f(x) = kx^2(1-x)$, $0 \leq x \leq 1$, be a probability density function?

(b) Find $\Pr(X \leq 1/2)$ for a random variable X with probability density function $f(x)$ (as in part (a)).

(c) Find $E(X)$, the expected value of a random variable X with probability density function $f(x)$ (as in part (a)).

18 (a) Find the solution to the initial value problem

$$y' = \frac{t^3}{y^2}$$
$$y(0) = 1.$$

(b) Find the general solution to the differential equation

$$y' + \frac{2}{x}y = \frac{\sin(x^2)}{x} \qquad (x > 0).$$

19 (a) Compute the inverse of $A = \begin{bmatrix} 1 & 2 & 1 \\ 1 & 0 & -1 \\ 1 & 6 & 5 \end{bmatrix}$ if it exists. Explain your reasoning.

(b) Compute the inverse of $B = \begin{bmatrix} 0 & 1 & 1 \\ 1 & -1 & -1 \\ 1 & -2 & -1 \end{bmatrix}$ if it exists. Explain your reasoning.

20 Compute the integral $\displaystyle\int_0^\infty \frac{e^x}{(e^x+1)^2}\,dx.$

21 Compute each of the following integrals.

(a) $\int_0^1 x^2 e^x \, dx$ (b) $\int_0^\infty \frac{x}{(1+x^2)^2} \, dx$

22 The probability density function for the random variable X is given by

$$f(x) = \begin{cases} \frac{k}{\sqrt{x}} & \text{for } 0 < x < 4 \\ 0 & \text{elsewhere.} \end{cases}$$

(a) Find the value of k that makes a probability density function.

(b) Calculate $\Pr(X < 1/4)$.

23 Solve the initial value problem $yy' = t \sin(t^2 - 1)$ with $y(1) = 0$.

24 Find the general solution of $y' + \frac{3x}{x^2+1} y = \frac{6x}{x^2+1}$.

25 Various doses of a poisonous substance were given to groups of 25 mice and the following results were observed:

Dose (mg) (x)	4	6	8	10	12	14	16
Number of Deaths (y)	1	3	6	8	14	16	20

(a) Find the equation of least squares line to fit these data.

(b) Use (a) to estimate the number of deaths in a group of 25 mice who receive a 7-milligram dose of this substance.

26 In this problem you will find the area of the region bounded by $y = \frac{1}{x}$, $y = 4x$, and $y = \frac{x}{2}$ for $x \geq 0$. Use the following steps:

(a) Sketch a rough picture of the region and find the intersection points of the curves.

(b) Express the area of the region by integrals, and compute it.

27 In the game Roulette, the chances of the ball landing on a green number are $1/19$. What is the probability that we do not see the ball land on a green number for the first 3 spins of the roulette wheel?

28 Let Z be the standard normal random variable (that is, mean $\mu = 0$ and standard deviation $\sigma = 1$). Show that the expected value $E(Z) = 0$.

Hint: $\int_{-\infty}^{\infty} xf(x) \, dx = \int_{-\infty}^{0} xf(x) \, dx + \int_0^\infty xf(x) \, dx$

29 Suppose that the population of a city has a growth rate that is proportional to the population, P, at any time t.

(a) Write down a differential equation for P that models this growth pattern.

(b) Solve the differential equation and find P given that $P(0) = 100$ and $P(5) = 500$.

30 Find the minimum of $f(x, y, z) = x^2 + y^2 + z^2$ subject to the constraint $x + 2y + 4z = 4$. (Here f is the square of the distance to the origin, and $x + 2y + 4z = 4$ is a plane. Thus we're asking you to find the smallest (square of the) distance to the origin. The point at which this occurs is the point on the plane closest to the origin.)

31 In a small mountain town, you are told that the elevation of a point x thousand feet east and y thousand feet north of the center of town is given by

$$z = f(x, y) = 3xy - x^2 - 2y^2 + 8 \text{ thousand feet.}$$

(In particular, when $x = -1$, the point is 1 thousand feet *west* of the center of town.)

(a) Show that the center of town $(x, y) = (0, 0)$ is a critical point of the function $z = f(x, y)$.

(b) Is the center of town a local maximum, a local minimum, or a saddle point?

(c) A particularly compulsive resident wants to live exactly 4 thousand feet from the center of town, and as high (elevation) as possible. Where should this person live?

32 For each of the following functions, find the critical points and classify them (if possible) as local maxima, local minima, or saddle points.

(a) $f(x, y) = 3x^2 + 2y^2 + 4x - 5y + 55$

(b) $f(x, y) = x^2 - 2y^2 + 8xy - 6x + 3y + 16$

(c) $f(x, y) = x^3 + 3y^2 - \dfrac{3}{2}x^2y + 12$

33 A study of births in Fredonia shows that birth weight is essentially normally distributed with mean $\mu = 3750$ g and standard deviation $\sigma = 525$ g. For our purposes we will assume that the birth weight is normally distributed.

(a) Find the probability that a randomly selected baby in Fredonia weighed between 3000 and 4000 grams at birth.

(b) Find the "middle 80%" of Fredonia birth weights. That is, find a range of weights, centered around the mean, so that 80% of babies weigh in this range at birth.

34 Kevin's company wishes to set aside funds for future expansion and so arranges to make continuous deposits into a bank account at the rate of \$20,000 per year. The bank account earns 4% interest compounded continuously.

(a) Set up the differential equation that is satisfied by the amount $f(t)$ of money in the account at time t.

(b) Solve the differential equation in part (a), assuming that $f(0) = 0$, and determine how much money will be be in the account at the end of 3 years.