31 In a small mountain town, you are told that the elevation of a point x thousand feet east and y thousand feet north of the center of town is given by

$$z = f(x, y) = 3xy - x^2 - 2y^2 + 8 \text{ thousand feet.}$$

(In particular, when $x = -1$, the point is 1 thousand feet *west* of the center of town.)

(a) Show that the center of town $(x, y) = (0, 0)$ is a critical point of the function $z = f(x, y)$.
Solution: This asks to show that $f_x = 0$ and $f_y = 0$ at $(x, y) = (0, 0)$. We compute these derivatives:

$$f_x = 3y - 2x \qquad \text{and} \qquad f_y = 3x - 4y.$$

It should be clear that both of these functions are zero when $x = 0$ and $y = 0$.

(b) Is the center of town a local maximum, a local minimum, or a saddle point?
Solution: This requires that we compute the discriminant $D = f_{xx}f_{yy} - f_{xy}^2$ at the origin $(0, 0)$. We compute the second derivatives:

$$f_{xx} = -2 \qquad f_{xy} = f_{yx} = 3 \qquad \text{and} \qquad f_{yy} = -4.$$

Thus $D = f_{xx}f_{yy} - f_{xy}^2 = (-2)(-4) - 3^2 = 8 - 9 = -1$. Since $D < 0$ at the critical point $(x, y) = (0, 0)$, this critical point is a saddle point.

(c) A particularly compulsive resident wants to live exactly 4 thousand feet from the center of town, and as high (elevation) as possible. Where should this person live?
Solution: This is hard, it turns out. This is supposed to be a Lagrange multipliers problem, with the constraint $g(x, y) = x^2 + y^2 - 4^2 = 0$. (This is a circle of radius 4 thousand feet.) As usual for Lagrange multiplier problems, we form the new function

$$F(x, y, \lambda) = f(x, y) + \lambda g(x, y) = 3xy - x^2 - 2y^2 + 8 + \lambda(x^2 + y^2 - 16)$$

then find its critical points. That is, we solve the equations

$$\begin{aligned} F_x &= 0 & &\text{or} & 3y - 2x + \lambda(2x) &= 0 \\ F_y &= 0 & & & 3x - 4y + \lambda(2y) &= 0 \\ F_\lambda &= 0 & & & x^2 + y^2 - 16 &= 0 \end{aligned}$$

Solving these equations is difficult, so I won't do it. Here's a tip for a method that is accessible to this class: divide the first two equations by x to get two equations in two variables: λ and $w = {}^{y}\!/_{x}$. (At the end, you'll have to justify that $x = 0$ is not a maximum for $f(x, y)$.) Now we can solve for w, which it turns out is

$$w = \frac{y}{x} = \frac{-1 \pm \sqrt{10}}{3}.$$

(This is also $\tan(\arctan(3)/2)$, if that suggests anything to you.) Now solve for y in terms of x and substitute this into the constraint equation $x^2 + y^2 = 16$. You should get four points: two maxima and two minima. (What we're doing there is saying that solving for w gives us two lines $y = kx$, and we see where those lines intersect the circle $x^2 + y^2 = 16$.)

32 For each of the following functions, find the critical points and classify them (if possible) as local maxima, local minima, or saddle points.

(a) $f(x, y) = 3x^2 + 2y^2 + 4x - 5y + 55$
Solution: We compute our first derivatives:

$$f_x = 6x + 4 \qquad \text{and} \qquad f_y = 4y - 5.$$

Our critical point (where $f_x = 0$ and $f_y = 0$) is therefore $(x, y) = (-2/3, 5/4)$. To classify this point, we need the second derivatives:

$$f_{xx} = 6 \qquad f_{xy} = f_{yx} = 0 \qquad \text{and} \qquad f_{yy} = 4.$$

Thus $D = f_{xx}f_{yy} - f_{xy}^2 = (6)(4) - 0^2 = 24 > 0$. Since $D > 0$ and $f_{xx} > 0$ at the critical point, this critical point is a local minimum.

(b) $f(x, y) = x^2 - 2y^2 + 8xy - 6x + 3y + 16$

Solution: We compute our first derivatives:

$$f_x = 2x + 8y - 6 \qquad \text{and} \qquad f_y = -4y + 8x + 3$$

Our critical point (where $f_x = 0$ and $f_y = 0$) is therefore $(x, y) = (0, {}^3\!/_4)$. To classify this point, we need the second derivatives:

$$f_{xx} = 2 \qquad f_{xy} = f_{yx} = 8 \qquad \text{and} \qquad f_{yy} = -4.$$

Thus $D = f_{xx}f_{yy} - f_{xy}^2 = (2)(-4) - 8^2 = -72 < 0$. Since $D < 0$ at the critical point, this critical point is a saddle point.

(c) $f(x, y) = x^3 + 3y^2 - \dfrac{3}{2}x^2y + 12$

Solution: We compute our first derivatives:

$$f_x = 3x^2 - 3xy \qquad \text{and} \qquad f_y = 6y - \frac{3}{2}x^2.$$

Our critical points (where $f_x = 0$ and $f_y = 0$) are therefore $(x, y) = (0, 0)$ or $(x, y) = (4, 4)$. To classify these points, we need the second derivatives:

$$f_{xx} = 6x - 3y \qquad f_{xy} = f_{yx} = -3x \qquad \text{and} \qquad f_{yy} = 6.$$

Thus $D = f_{xx}f_{yy} - f_{xy}^2 = (6x - 3y)(6) - (-3x)^2 = 36x - 18y - 9x^2$. At $(x, y) = (4, 4)$, we get $D = 36(4) - 18(4) - 9(4^2) = -72 < 0$. Thus $(4, 4)$ is a saddle point. At $(x, y) = (0, 0)$, we get $D = 0$. Thus this test fails, and we don't know what happens at this critical point.

33 A study of births in Fredonia shows that birth weight is essentially normally distributed with mean $\mu = 3750$ g and standard deviation $\sigma = 525$ g. For our purposes we will assume that the birth weight is normally distributed.

(a) Find the probability that a randomly selected baby in Fredonia weighed between 3000 and 4000 grams at birth.

Solution: This asks for the probability $\Pr(3000 \leq X \leq 4000)$, where X is a normal distribution with mean $\mu = 3750$ and standard deviation $\sigma = 525$. We turn this into a question about the standard normal distribution Z in the usual way:

$$\begin{aligned}\Pr(3000 \leq X \leq 4000) &= \Pr\left(\frac{3000 - \mu}{\sigma} \leq Z \leq \frac{4000 - \mu}{\sigma}\right) \\ &= \Pr\left(\frac{3000 - 3750}{525} \leq Z \leq \frac{4000 - 3750}{525}\right) \\ &\approx \Pr(-1.43 \leq Z \leq 0.48).\end{aligned}$$

(There has been some rounding – more about this later.) Now we exploit the symmetry of the normal distribution to turn this probability into two probabilities we can find on our table:

$$\begin{aligned}\Pr(3000 \leq X \leq 4000) &\approx \Pr(-1.43 \leq Z \leq 0.48) \\ &= \Pr(-1.43 \leq Z \leq 0) + \Pr(0 \leq Z \leq 0.48) \\ &= \Pr(0 \leq Z \leq 1.43) + \Pr(0 \leq Z \leq 0.48) \\ &= 0.4236 + 0.1844 = 0.6080.\end{aligned}$$

That is, the probability that a randomly selected baby in Fredonia weighed between 3000 and 4000 grams at birth is about 60.80%.

Now a comment on rounding. My TI-83 says the answer is

$$\texttt{normalcdf}(3000, 4000, 3750, 525) = .6064668803,$$

whereas Mathematica (which I trust more) says that this is roughly 0.60646693. We've gone wrong by rounding up twice: we rounded 1.428571429 to 1.43 and 0.4761904762 to 0.48. We could try to get a better estimate by interpolating the table values, but it seems simpler to be aware of the inaccuracies of our assumptions.

(b) Find the "middle 80%" of Fredonia birth weights. That is, find a range of weights, centered around the mean, so that 80% of babies weigh in this range at birth.

Solution: We would like to find a value k so that $\Pr(\mu - k \le X \le \mu + k) = 0.80$. As usual, we turn this into a question about the standard normal distribution Z:

$$0.80 = \Pr(\mu - k \le X \le \mu + k) = \Pr\left(\frac{\mu - k - \mu}{\sigma} \le Z \le \frac{\mu + k - \mu}{\sigma}\right) = \Pr\left(-\frac{k}{\sigma} \le Z \le \frac{k}{\sigma}\right).$$

Thus we're really trying to find a value z so that $\Pr(-z \le Z \le z) = 0.80$ (and $z = \frac{k}{\sigma}$). Using the symmetry of the standard normal distribution, we get that

$$0.80 = \Pr(-z \le Z \le z) = \Pr(-z \le Z \le 0) + \Pr(0 \le Z \le z) = 2\Pr(0 \le Z \le z).$$

Thus we're looking for the value of z for which $\Pr(0 \le Z \le z) = 0.40$. Alas, nothing on the table gives us a Z-value of precisely 0.4000. We can interpolate, however, to find such a better approximation of such a z:

z	1.28	z	1.29
Z-value	0.3997	0.4000	0.4015

Often what's done in this situation is we interpolate between 1.28 and 1.29 to find the z with Z-value 0.4000:

$$\frac{\Delta z}{\Delta \text{ Value}} = \frac{z - 1.28}{0.4000 - 0.3997} = \frac{1.29 - 1.28}{0.4015 - 0.3997} \quad \text{or} \quad \frac{z - 1.28}{0.0003} = \frac{0.01}{0.0018}.$$

Thus $z = 1.28 + \frac{0.01}{0.0018}(0.0003) \approx 1.282$.
Solving for k in $z = \frac{k}{\sigma} = \frac{k}{525}$, we get the following possible values:

z	x
1.28	672
1.282	673.05
1.29	677.25

So roughly speaking the answer is always around 675 g; that is, the middle 80% of Fredonia babies weigh in the range 3075 to 4425 grams.

34 Kevin's company wishes to set aside funds for future expansion and so arranges to make continuous deposits into a bank account at the rate of \$20,000 per year. The bank account earns 4% interest compounded continuously.

(a) Set up the differential equation that is satisfied by the amount $f(t)$ of money in the account at time t.

Solution: The amount of money $f(t)$ changes in two ways: there is a continuous deposit of \$20,000 per year, and there is a bank account that earns 4% interest compounded continuously. Thus

$$f'(t) = \underbrace{20{,}000}_{\text{deposit}} + \underbrace{0.04 f(t)}_{\text{interest}}$$

is the required differential equation.

(b) Solve the differential equation in part (a), assuming that $f(0) = 0$, and determine how much money will be be in the account at the end of 3 years.

Solution: We re-write the differential equation as

$$f'(t) - 0.04f(t) = 20,000. \qquad (**)$$

This is a first-order linear differential equation (it is of the form $y' + P(t)y = G(t)$, where $P(t) = -0.04$ and $G(t) = 20,000$). Then $\int P(t)\,dt = -0.04t$ (we have chosen $K = 0$ as we need only *an* anti-derivative, not *the* anti-derivative). The integrating factor is then $h(t) = e^{\int P(t)\,dt} = e^{-0.04t}$, and when we multiply equation $(**)$ through by this factor, we get

$$f'(t)e^{-0.04t} - 0.04f(t)e^{-0.04t} = 20,000e^{-0.04t}$$

or

$$\left(f(t)e^{-0.04t}\right)' = 20,000e^{-0.04t}.$$

Now we integrate both sides of this equation to get

$$\begin{aligned} f(t)e^{-0.04t} &= 20,000 \cdot \frac{1}{-0.04}e^{-0.04t} + K \\ &= -500,000e^{-0.04t} + K. \end{aligned}$$

Multiply this by $e^{0.04t}$ to solve for $f(t)$:

$$f(t) = -500,000 + Ke^{0.04t}.$$

Now the initial condition $f(0) = 0$ will allow us to solve for K:

$$0 = -500,000 + Ke^{0.04(0)} = -500,000 + K.$$

Thus $K = 500,000$ and $f(t) = -500,000 + 500,000e^{0.04t} = 500,000(e^{0.04t} - 1)$.

Finally, we are asked how much money will be in the account at the end of 3 years. This is $f(3) = 500,000(e^{0.04(3)} - 1) \approx \$63,748.43$. (This number is at least in the right neighborhood – we've contributed \$60,000 to this account over these 3 years.)