**25** Various doses of a poisonous substance were given to groups of 25 mice and the following results were observed:

| Dose (mg) ($x$) | 4 | 6 | 8 | 10 | 12 | 14 | 16 |
|---|---|---|---|---|---|---|---|
| Number of Deaths ($y$) | 1 | 3 | 6 | 8 | 14 | 16 | 20 |

(a) Find the equation of least squares line to fit these data.

**Solution:** As with previous least squares problems, there are two approaches to finding the best fit line $y = mx + b$. The first is the linear algebra approach, which involves looking at the system $A^T AX = A^T B$, where

$$A = \begin{bmatrix} x_1 & 1 \\ x_2 & 1 \\ \vdots & \vdots \\ x_n & 1 \end{bmatrix} = \begin{bmatrix} 4 & 1 \\ 6 & 1 \\ 8 & 1 \\ 10 & 1 \\ 12 & 1 \\ 14 & 1 \\ 16 & 1 \end{bmatrix}, \qquad X = \begin{bmatrix} m \\ b \end{bmatrix}, \qquad \text{and} \qquad Y = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \\ 6 \\ 8 \\ 14 \\ 16 \\ 20 \end{bmatrix}.$$

Then the linear system $A^T AX = A^T B$ is

$$\begin{bmatrix} 4 & 6 & 8 & 10 & 12 & 14 & 16 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 4 & 1 \\ 6 & 1 \\ 8 & 1 \\ 10 & 1 \\ 12 & 1 \\ 14 & 1 \\ 16 & 1 \end{bmatrix} \begin{bmatrix} m \\ b \end{bmatrix} = \begin{bmatrix} 4 & 6 & 8 & 10 & 12 & 14 & 16 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 3 \\ 6 \\ 8 \\ 14 \\ 16 \\ 20 \end{bmatrix}.$$

or

$$\begin{bmatrix} 812 & 70 \\ 70 & 7 \end{bmatrix} \begin{bmatrix} m \\ b \end{bmatrix} = \begin{bmatrix} 862 \\ 68 \end{bmatrix}. \tag{1}$$

On the other hand, we could also use the partial derivatives approach, from which we've learned that the linear system to solve in order to find $m$ and $b$ is

$$\begin{bmatrix} \sum_{i=1}^{n} x_i^2 & \sum_{i=1}^{n} x_i \\ \sum_{i=1}^{n} x_i & n \end{bmatrix} \begin{bmatrix} m \\ b \end{bmatrix} = \begin{bmatrix} \sum_{i=1}^{n} x_i y_i \\ \sum_{i=1}^{n} y_i \end{bmatrix}.$$

Let's compute with the data:

| $i$ | $x_i$ | $y_i$ | $x_i^2$ | $x_i y_i$ |
|---|---|---|---|---|
| 1 | 4 | 1 | 16 | 4 |
| 2 | 6 | 3 | 36 | 18 |
| 3 | 8 | 6 | 64 | 48 |
| 4 | 10 | 8 | 100 | 64 |
| 5 | 12 | 14 | 144 | 168 |
| 6 | 14 | 16 | 196 | 224 |
| 7 | 16 | 20 | 256 | 320 |
| Sum: | 70 | 68 | 812 | 862 |

We can then see the linear system is precisely that shown in equation (1). We solve this system to see that

$$m = \frac{13}{8} = 1.625 \qquad \text{and} \qquad b = -\frac{183}{28} \approx -6.535714286$$

Thus the least-squares line is $y = \frac{13}{8}x - \frac{183}{28}$.

(b) Use (a) to estimate the number of deaths in a group of 25 mice who receive a 7-milligram dose of this substance.

**Solution:** This question asks for $y$, the number of deaths, if $x$, the dosage, is 7 milligrams. This is simply

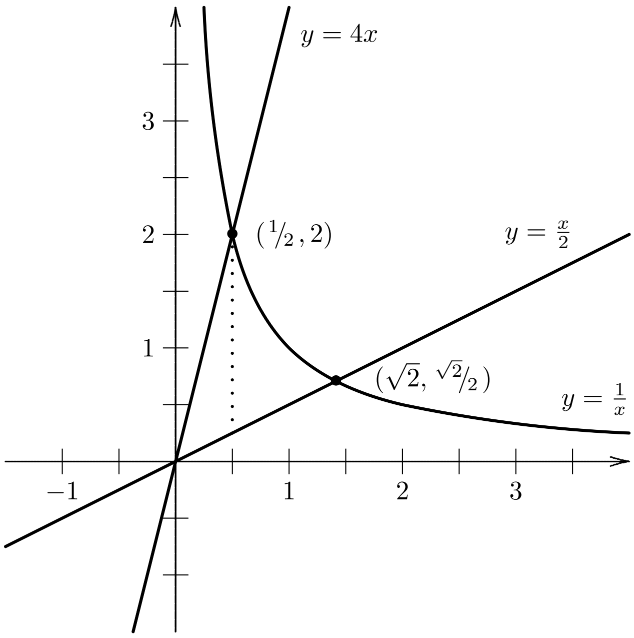
$$y = \frac{13}{8} \cdot 7 - \frac{183}{28} = \frac{271}{56} \approx 4.839286.$$

That is, there will be 4 or 5 deaths (according to our model) with a dosage of 7 milligrams.

26 In this problem you will find the area of the region bounded by $y = \frac{1}{x}$, $y = 4x$, and $y = \frac{x}{2}$ for $x \geq 0$. Use the following steps:

(a) Sketch a rough picture of the region and find the intersection points of the curves.

**Solution:** Here is a picture of the situation:




(b) Express the area of the region by integrals, and compute it.

**Solution:** In general, the area between two curves (from $x = a$ to $x = b$) is given by

$$\text{Area} = \int_a^b (\text{ top curve } - \text{ bottom curve }) \, dx.$$

In this case, the bottom curve is $y = x/2$, but the top curve is either $y = 4x$ or $y = 1/x$. These two top curves cross when $4x = 1/x$, which happens at $x = 1/2$ (assuming $x \geq 0$). Thus the left-hand part of the region (to the left of the dotted line) has area

$$\text{Area on Left} = A_L = \int_0^{1/2} \left(4x - \frac{x}{2}\right) dx = \int_0^{1/2} \frac{7}{2}x \, dx = \frac{7}{4}x^2 \Big|_0^{1/2} = \frac{7}{4}\left(\left(\frac{1}{2}\right)^2 - 0^2\right) = \frac{7}{16} = 0.4375.$$

Now we turn to the area on the right, which is bounded by the dotted line $x = 1/2$ on the left, $y = 1/x$ on the top, and $y = x/2$ on the bottom. These last two curves cross when $1/x = x/2$ (and $x \geq 0$), or at

$x = \sqrt{2}$. Thus the area on the right is

$$\begin{aligned} \text{Area on Right} = A_R &= \int_{1/2}^{\sqrt{2}} \left( \frac{1}{x} - \frac{x}{2} \right) \, dx \\ &= \left( \ln(x) - \frac{1}{4}x^2 \right) \Big|_{1/2}^{\sqrt{2}} \\ &= \left( \ln(\sqrt{2}) - \frac{1}{4}(\sqrt{2})^2 \right) - \left( \ln\left(1/2\right) - \frac{1}{4}\left(1/2\right) \right) \\ &= \ln(2^{1/2}) - \frac{1}{2} - \ln(2^{-1}) + \frac{1}{16} \\ &= \frac{1}{2}\ln(2) - \frac{7}{16} + \ln(2) = \frac{3}{2}\ln(2) - \frac{7}{16} \end{aligned}$$

Thus the total area is $A_L + A_R = \frac{3}{2}\ln(2) \approx 1.03972$.

27 In the game Roulette, the chances of the ball landing on a green number are $1/19$. What is the probability that we do not see the ball land on a green number for the first 3 spins of the roulette wheel? (Hint: Define a geometric random variable.)

**Solution:** Let $G$ be the event "the ball lands on green." We're asked for $\Pr(\overline{G}_1 \cap \overline{G}_2 \cap \overline{G}_3)$, the probability that the ball *doesn't* land on green for the first three spins. Since $\Pr(G) = 1/19$, we must have $\Pr(\overline{G}) = 18/19$. Since the spins are independent (one spin has no influence over any other), we get

$$\Pr(\overline{G}_1 \cap \overline{G}_2 \cap \overline{G}_3) = \Pr(\overline{G}_1) \cdot \Pr(\overline{G}_2) \cdot \Pr(\overline{G}_3) = \left( \frac{18}{19} \right)^3 \approx 0.8502697,$$

or about 85.03%.

Another approach is to look at this as a Bernoulli trial. We have $n = 3$ trials. If we define "success" to be the ball landing on green, then we're looking for the probability of $k = 0$ successes, where the probability of success is $p = 1/19$. Thus the probability of the ball not landing on green is

$$\Pr(\text{ no green }) = \Pr(\text{ 0 "successes" }) = b(3, 0; 1/19) = \binom{3}{0} \left( \frac{1}{19} \right)^0 \left( \frac{18}{19} \right)^3 = \left( \frac{18}{19} \right)^3,$$

as before.

28 Let $Z$ be the standard normal random variable (that is, mean $\mu = 0$ and standard deviation $\sigma = 1$). Show that the expected value $E(Z) = 0$.

Hint: $\displaystyle\int_{-\infty}^{\infty} x f(x) \, dx = \int_{-\infty}^{0} x f(x) \, dx + \int_{0}^{\infty} x f(x) \, dx$

**Solution:** The standard normal density function is $f(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2}$, defined for all $x$ (that is, for $-\infty < x < \infty$). We are thus asked to calculate the expected value, which is

$$E(X) = \int_{-\infty}^{\infty} x f(x) \, dx = \int_{-\infty}^{\infty} x \cdot \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2} \, dx. = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} x e^{-\frac{1}{2}x^2} \, dx.$$

Rather than jumping in and computing the (doubly!) improper integral, let's first compute the indefinite integral using the substitution $u = -\frac{1}{2}x^2$ (so $du = -x \, dx$):

$$\int x e^{-\frac{1}{2}x^2} \, dx = -\int e^{-\frac{1}{2}x^2} \cdot -x \, dx = -\int e^u \, du = -e^u + K = -e^{-\frac{1}{2}x^2} + K.$$

Thus, using the hint, we compute the expected value:

$$\begin{aligned}
E(X) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} xe^{-\frac{1}{2}x^2}\, dx \\
&= \frac{1}{\sqrt{2\pi}} \left[ \int_{-\infty}^{0} xe^{-\frac{1}{2}x^2}\, dx + \int_{0}^{\infty} xe^{-\frac{1}{2}x^2}\, dx \right] \\
&= \frac{1}{\sqrt{2\pi}} \left[ \lim_{a\to-\infty} \int_{a}^{0} xe^{-\frac{1}{2}x^2}\, dx + \lim_{b\to\infty} \int_{0}^{b} xe^{-\frac{1}{2}x^2}\, dx \right] \\
&= \frac{1}{\sqrt{2\pi}} \left[ \lim_{a\to-\infty} \left( -e^{-\frac{1}{2}x^2} \Big|_{a}^{0} \right) + \lim_{b\to\infty} \left( -e^{-\frac{1}{2}x^2} \Big|_{0}^{b} \right) \right] \\
&= \frac{1}{\sqrt{2\pi}} \left[ \lim_{a\to-\infty} \left( -e^{0} + e^{-\frac{1}{2}a^2} \right) + \lim_{b\to\infty} \left( -e^{-\frac{1}{2}b^2} + e^{0} \right) \right] \\
&= \frac{1}{\sqrt{2\pi}} \left[ (-1+0) + (-0+1) \right] \\
&= 0.
\end{aligned}$$

(Here we've used the fact that

$$\lim_{a\to-\infty} e^{-\frac{1}{2}a^2} = \lim_{b\to\infty} e^{-\frac{1}{2}b^2} = 0.)$$

Thus the expected value (or mean) is zero, as we should expect.

**29** Suppose that the population of a city has a growth rate that is proportional to the population, $P$, at any time $t$.

(a) Write down a differential equation for $P$ that models this growth pattern.

**Solution:** The growth rate is the rate of change in the population, or $P'(t)$. We're told that this is proportional to the population $P(t)$. This means that, for some constant $k$,

$$P'(t) = kP(t).$$

This is our differential equation.

(b) Solve the differential equation and find $P$ given that $P(0) = 100$ and $P(5) = 500$.

**Solution:** The differential equation from (a) is separable. Write $P'(t)$ as $\frac{dP}{dt}$, so the equation looks like

$$\frac{dP}{dt} = kP \qquad \text{or} \qquad \frac{dP}{P} = k\, dt.$$

Now integrate to get $\ln(|P|) = kt + K$. Exponentiate both sides:

$$e^{\ln(|P|)} = e^{kt+K} = e^{kt} \cdot e^{K} \qquad \text{or} \qquad P = Ce^{kt}.$$

The first initial condition says that $P = 100$ when $t = 0$. This means that $100 = Ce^{k\cdot 0} = Ce^0 = C$. Thus we have $P = 100e^{kt}$. The second initial condition will allow us to find $k$: $P = 500$ when $t = 5$. Thus $500 = 100e^{k\cdot 5}$ or $e^{5k} = \frac{500}{100} = 5$. Taking the natural logarithm of both sides gives $5k = \ln(e^{5k}) = \ln(5)$, so $k = \frac{\ln(5)}{5} \approx 0.3218875825$. Thus our model for the population at time $t$ is

$$P(t) = 100e^{\frac{\ln(5)}{5}t}.$$

**30** Find the minimum of $f(x, y, z) = x^2 + y^2 + z^2$ subject to the constraint $x + 2y + 4z = 4$. (Here $f$ is the square of the distance to the origin, and $x + 2y + 4z = 4$ is a plane. Thus we're asking you to find the smallest (square of the) distance to the origin. The point at which this occurs is the point on the plane closest to the origin.)

**Solution:** We're trying to minimize the function $f(x, y, z) = x^2 + y^2 + z^2$ subject to the constraint $g(x, y, z) = x + 2y + 4z - 4 = 0$. This calls for Lagrange multipliers: form the new function

$$F(x, y, z, \lambda) = f(x, y, z) + \lambda g(x, y, z) = x^2 + y^2 + z^2 + \lambda(x + 2y + 4z - 4).$$

We then find the critical points of this new function $F$. That is, we solve the equations

$$\begin{aligned} F_x &= 0 & \text{or} \qquad 2x + \lambda(1) &= 0 \\ F_y &= 0 & 2y + \lambda(2) &= 0 \\ F_z &= 0 & 2z + \lambda(4) &= 0 \\ F_\lambda &= 0 & x + 2y + 4z - 4 &= 0 \end{aligned}$$

From the first three equations, we solve for $-\lambda$ in each one. This gives us $-\lambda = 2x = y = \frac{1}{2}z$. Thus we can write $y$ and $z$ in terms of $x$: $y = 2x$ and $z = 4x$. Plug these into the final equation (the constraint equation) to get

$$4 = x + 2y + 4z = x + 2(2x) + 4(4x) = 21x.$$

Thus $x = 4/21$, $y = 2x = 8/21$, and $z = 16/21$. Some thought will convince you that this point $(x, y, z) = (4/21, 8/21, 16/21)$ must be a minimum point of $f(x, y, z)$.