19 (a) Compute the inverse of $A = \begin{bmatrix} 1 & 2 & 1 \\ 1 & 0 & -1 \\ 1 & 6 & 5 \end{bmatrix}$ as if it exists. Explain your reasoning.

Solution: The simplest way to do this is to row reduce $[A \mid I\,]$ into $[\,I \mid A^{-1}]$, then read the inverse matrix from this form.

We perform this calculation:

$$[A \mid I\,] = \left[\begin{array}{ccc|ccc} 1 & 2 & 1 & 1 & 0 & 0 \\ 1 & 0 & -1 & 0 & 1 & 0 \\ 1 & 6 & 5 & 0 & 0 & 1 \end{array}\right]$$

$$\longrightarrow \left[\begin{array}{ccc|ccc} 1 & 2 & 1 & 1 & 0 & 0 \\ 0 & -2 & -2 & -1 & 1 & 0 \\ 0 & 4 & 4 & -1 & 0 & 1 \end{array}\right] \qquad \begin{array}{l} R_2 = r_2 - r_1 \\ R_3 = r_3 - r_1 \end{array}$$

$$\longrightarrow \left[\begin{array}{ccc|ccc} 1 & 2 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1/2 & -1/2 & 0 \\ 0 & 4 & 4 & -1 & 0 & 1 \end{array}\right] \qquad R_2 = -\tfrac{1}{2}r_2)$$

$$\longrightarrow \left[\begin{array}{ccc|ccc} 1 & 2 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1/2 & -1/2 & 0 \\ 0 & 0 & 0 & -3 & 2 & 1 \end{array}\right] \qquad R_3 = r_3 - 4r_2.$$

Now we're stymied by this row of zeros to the left of the vertical bar. We can't possibly reduce this to $[\,I \mid A^{-1}\,]$, and this means that there is no inverse of A. (Notice that every step we've done is reversible, so we haven't lost any information at any step.)

(b) Compute the inverse of $B = \begin{bmatrix} 0 & 1 & 1 \\ 1 & -1 & -1 \\ 1 & -2 & -1 \end{bmatrix}$ if it exists. Explain your reasoning.

Solution: The simplest way to do this is to row reduce $[B \mid I\,]$ into $[\,I \mid B^{-1}]$, then read the inverse matrix from this form.

We perform this calculation:

$$[B \mid I\,] = \left[\begin{array}{ccc|ccc} 0 & 1 & 1 & 1 & 0 & 0 \\ 1 & -1 & -1 & 0 & 1 & 0 \\ 1 & -2 & -1 & 0 & 0 & 1 \end{array}\right] \longrightarrow \left[\begin{array}{ccc|ccc} 1 & -1 & -1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 & 0 & 0 \\ 1 & -2 & -1 & 0 & 0 & 1 \end{array}\right] \qquad R_1 \leftrightarrow R_2$$

$$\longrightarrow \left[\begin{array}{ccc|ccc} 1 & -1 & -1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 & 0 & 0 \\ 0 & -1 & 0 & 0 & -1 & 1 \end{array}\right] \qquad R_3 = r_3 - r_1$$

$$\longrightarrow \left[\begin{array}{ccc|ccc} 1 & -1 & -1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & -1 & 1 \end{array}\right] \qquad R_3 = r_3 + r_2$$

$$\longrightarrow \left[\begin{array}{ccc|ccc} 1 & -1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 & -1 \\ 0 & 0 & 1 & 1 & -1 & 1 \end{array}\right] \qquad \begin{array}{l} R_1 = r_1 + r_3 \\ R_2 = r_2 - r_3 \end{array}$$

$$\longrightarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 & -1 \\ 0 & 0 & 1 & 1 & -1 & 1 \end{array}\right] \qquad R_1 = r_1 + r_2$$

$$\longrightarrow [\,I \mid B^{-1}\,].$$

Thus the inverse of B is $B^{-1} = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & -1 \\ 1 & -1 & 1 \end{bmatrix}$.

20 Compute the integral $\displaystyle\int_0^\infty \frac{e^x}{(e^x+1)^2}\,dx$.

Solution: In order to compute the indefinite integral $\displaystyle\int \frac{e^x}{(e^x+1)^2}\,dx$, we make the substitution $u = e^x + 1$. Then $du = u'\,dx = e^x\,dx$, or

$$\int \frac{e^x}{(e^x+1)^2}\,dx = \int \frac{du}{u^2} = \int u^{-2}\,du.$$

Integrating, this is $-u^{-1} + K = -\frac{1}{u} + K = -\frac{1}{e^x+1} + K$.
Thus, for the improper integral, we have

$$\begin{aligned}
\int_0^\infty \frac{e^x}{(e^x+1)^2}\,dx &= \lim_{b\to\infty} \int_0^b \frac{e^x}{(e^x+1)^2}\,dx \\
&= \lim_{b\to\infty} \left[-\frac{1}{e^x+1}\right]_0^b \\
&= \lim_{b\to\infty} \left[-\frac{1}{e^b+1} - \left(-\frac{1}{e^0+1}\right)\right] \\
&= -0 + \frac{1}{2}.
\end{aligned}$$

This is the value of our improper integral: $\displaystyle\int_0^\infty \frac{e^x}{(e^x+1)^2}\,dx = \frac{1}{2}$.

21 Compute each of the following integrals.

(a) $\displaystyle\int_0^1 x^2 e^x\,dx$

Solution: Let's begin by computing the indefinite integral $\int x^2 e^x\,dx$. We do this using the technique of integration by parts. Let

$$\begin{aligned}
u &= x^2 & dv &= e^x\,dx \\
du &= 2x\,dx & v &= e^x.
\end{aligned}$$

Plugging this into the integration by parts formula gives us

$$\int x^2 e^x\,dx = \int u\,dv = uv - \int v\,du = x^2 e^x - \int e^x \cdot 2x\,dx.$$

This is *still* an integral we can't compute directly, but it is simpler than the original integral. We use integration by parts again, this time with

$$\begin{aligned}
u &= x & dv &= e^x\,dx \\
du &= dx & v &= e^x.
\end{aligned}$$

Then

$$\begin{aligned}
\int x^2 e^x\,dx &= x^2 e^x - 2\int x e^x\,dx \\
&= x^2 e^x - 2\left(x e^x - \int e^x\,dx\right) \\
&= x^2 e^x - 2\left(x e^x - e^x + K\right) \\
&= x^2 e^x - 2x e^x + 2e^x + C \\
&= \left(x^2 - 2x + 2\right) e^x + C.
\end{aligned}$$

Thus the definite integral may be calculated as

$$\begin{aligned}\int_0^1 x^2e^x\,dx &= \left(x^2-2x+2\right)e^x\Big|_0^1\\ &= \left(1^2-2(1)+2\right)e^1-\left(0^2-2(0)+2\right)e^0\\ &= e-2\approx 0.71828183.\end{aligned}$$

(b) $\displaystyle\int_0^\infty \frac{x}{\left(1+x^2\right)^2}\,dx$

Solution: We again compute the indefinite integral first. Let's make the substitution

$$u=1+x^2 \qquad \text{so} \qquad du=2x\,dx \qquad \text{or} \qquad x\,dx=\frac{1}{2}du.$$

Plugging this in, we get

$$\begin{aligned}\int \frac{x}{\left(1+x^2\right)^2}\,dx &= \int \frac{\frac{1}{2}du}{u^2}\\ &= \frac{1}{2}\int u^{-2}\,du\\ &= \frac{1}{2}\cdot\frac{1}{-1}u^{-1}+K\\ &= -\frac{1}{2}\cdot\frac{1}{1+x^2}+K.\end{aligned}$$

Thus the definite (but improper) integral is

$$\begin{aligned}\int_0^\infty \frac{x}{\left(1+x^2\right)^2}\,dx &= \lim_{b\to\infty}\int_0^b \frac{x}{\left(1+x^2\right)^2}\,dx\\ &= \lim_{b\to\infty}\left(-\frac{1}{2}\cdot\frac{1}{1+x^2}\right)\Bigg|_0^b\\ &= \lim_{b\to\infty}\left[-\frac{1}{2}\cdot\frac{1}{1+b^2}-\left(-\frac{1}{2}\cdot\frac{1}{1+0^2}\right)\right]\\ &= 0+\frac{1}{2}.\end{aligned}$$

Thus the value of the integral is $\frac{1}{2}$.

22 The probability density function for the random variable X is given by

$$f(x)=\begin{cases}\dfrac{k}{\sqrt{x}} & \text{for } 0<x<4\\ 0 & \text{elsewhere.}\end{cases}$$

(a) Find the value of that makes a probability density function.

Solution: The idea here is that the total probability must be 1. That is, the integral of $f(x)$ over its entire domain is 1:

$$\int_0^4 \frac{k}{\sqrt{x}}\,dx=1.$$

We compute this integral to find k:

$$1=\int_0^4 kx^{-1/2}\,dx=k\frac{1}{1/2}x^{1/2}\Big|_0^4=2k\sqrt{x}\Big|_0^4=2k\left(\sqrt{4}-\sqrt{0}\right)=4k.$$

Thus $1=4k$, or $k=1/4$.

(b) Calculate $\Pr\left(X < {}^1\!/_4\right)$.

Solution: This is simply a computation, with one observation. This is that the probability that $X < {}^1\!/_4$ is the integral up to ${}^1\!/_4$. Since $f(x)$ is only positive from $x = 0$ to $x = 4$, this probability is the integral from $x = 0$ to $x = {}^1\!/_4$:

$$\begin{aligned}\Pr\left(X < {}^1\!/_4\right) &= \int_0^{1/4} \frac{1}{4}x^{-1/2}\,dx \\ &= \frac{1}{2}\sqrt{x}\Big|_0^{1/4} \\ &= \frac{1}{2}\left(\sqrt{\frac{1}{2}} - \sqrt{0}\right) \\ &= \frac{1}{4}.\end{aligned}$$

Thus this probability is ${}^1\!/_4$, or 25%.

23 Solve the initial value problem $yy' = t\sin\left(t^2-1\right)$ with $y(1) = 0$.

Solution: This is a separable differential equation. Let's write $y' = \frac{dy}{dt}$ and separate the y terms and the t terms:

$$y\frac{dy}{dt} = t\sin\left(t^2-1\right) \qquad \text{or} \qquad y\,dy = t\sin\left(t^2-1\right)\,dt.$$

Now we integrate both sides:

$$\int y\,dy = \int t\sin\left(t^2-1\right)\,dt.$$

The integral on the right-hand side requires a substitution $u = t^2 - 1$, so $du = 2t\,dt$ or $\frac{1}{2}\,du = t\,dt$. Thus

$$\int y\,dy = \int \sin\left(t^2-1\right)\cdot t\,dt = \int \sin(u)\cdot\frac{1}{2}\,du,$$

and so

$$\frac{1}{2}y^2 = -\frac{1}{2}\cos(u) + K \qquad \text{or} \qquad \frac{1}{2}y^2 = -\frac{1}{2}\cos(t^2-1) + K.$$

Solving for y, we get $y = \pm\sqrt{2K - \cos(t^2-1)}$. We use the initial condition ($y = 0$ when $t = 1$) to find K:

$$0 = \pm\sqrt{2K - \cos(1^2-1)} = \pm\sqrt{2K - \cos(0)} = \pm\sqrt{2K-1}.$$

Thus $2K = 1$, so $y = \pm\sqrt{1-\cos(t^2-1)}$.

Notice that we've really deduced two solutions here: $y = \sqrt{1-\cos(t^2-1)}$ and $y = -\sqrt{1-\cos(t^2-1)}$. Usually there is only one solution. This case is different because the initial condition (that $y = 0$) doesn't dictate a choice of sign.

24 Find the general solution of $y' + \dfrac{3x}{x^2+1}y = \dfrac{6x}{x^2+1}$.

Solution: This is a first-order linear differential equation in the standard form

$$y' + P(x)y = G(x).$$

Thus, in our case, $P(x) = \frac{3x}{x^2+1}$ and $G(x) = \frac{6x}{x^2+1}$. We solve this by multiplying the equation by the integrating factor $h(x) = e^{\int P(x)\,dx}$. This integral requires some work (via the substitution $u = x^2+1$, so $du = 2x\,dx$):

$$\int P(x)\,dx = \int \frac{3x}{x^2+1}\,dx = \frac{3}{2}\int\frac{2x\,dx}{x^2+1} = \frac{3}{2}\int\frac{du}{u} = \frac{3}{2}\ln(u) = \frac{3}{2}\ln(x^2+1) = \ln\left((x^2+1)^{3/2}\right).$$

We've written it like this so the integrating factor $h(x)$ will be easier to compute:

$$h(x) = e^{\int P(x)\,dx} = e^{\ln((x^2+1)^{3/2})} = (x^2+1)^{3/2}.$$

We multiply the differential equation by this integrating factor, and the real trick is that the left-hand side of the product is a derivative:

$$(x^2+1)^{3/2} \cdot \left(y' + \frac{3x}{x^2+1}y = \frac{6x}{x^2+1}\right) \qquad \text{or} \qquad \left((x^2+1)^{3/2}y\right)' = (x^2+1)^{3/2} \cdot \frac{6x}{x^2+1} = 6x(x^2+1)^{1/2}.$$

Now we integrate both sides of this equation:

$$(x^2+1)^{3/2}y = \int 6x(x^2+1)^{1/2}\,dx.$$

This integral requires the same substitution as before (namely, $u = x^2+1$, $du = 2x\,dx$, and so on):

$$(x^2+1)^{3/2}y = \int 6x(x^2+1)^{1/2}\,dx = \int 3(x^2+1)^{1/2} \cdot 2x\,dx = 3\int (u)^{1/2}du = 3\frac{1}{3/2}(u)^{3/2} + K = 2(x^2+1)^{3/2} + K.$$

We solve for y by dividing by $(x^2+1)^{3/2}$ and get the general solution

$$y = 2 + \frac{K}{(x^2+1)^{3/2}}.$$

This is our final answer.

(Notice that this differential is also separable, if you want to work at it. That is, you can write the equation as

$$\frac{dy}{dx} = \frac{6x}{x^2+1} - \frac{3x}{x^2+1}y = \frac{3x}{x^2+1}(2-y) \qquad \text{or} \qquad \frac{dy}{2-y} = \frac{3x}{x^2+1}\,dx.$$

Feel free to finish this computation.)