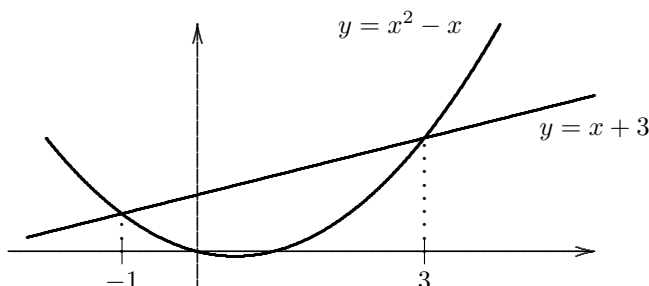


- 13 (a) Find the area of the region bounded by the curves $y = x^2 - x$ and $y = x + 3$.

Solution: These two curves intersect where $x^2 - x = x + 3$, or $x^2 - 2x - 3 = 0$. This factors to $(x - 3)(x + 1) = 0$, so the two intersection points are at $x = -1$ and $x = 3$. (Notice that between the two intersection points, the curve $y = x + 3$ is *above* the curve $y = x^2 - x$. (You can see this by plugging in a convenient value for x , say $x = 0$, and seeing that $0 + 3 > 0^2 - 0$, or by checking out the lovely picture, below.)



Thus the area of the region bounded by the two curves is

$$\begin{aligned}
 \text{Area} &= \int_{-1}^3 [(\text{top curve}) - (\text{bottom curve})] dx \\
 &= \int_{-1}^3 [(x + 3) - (x^2 - x)] dx \\
 &= \int_{-1}^3 (3 + 2x - x^2) dx \\
 &= \left[3x + x^2 - \frac{1}{3}x^3 \right]_{-1}^3 \\
 &= \left(3(3) + (3)^2 - \frac{1}{3}(3)^3 \right) - \left(3(-1) + (-1)^2 - \frac{1}{3}(-1)^3 \right) \\
 &= (9 + 9 - 9) - \left(-3 + 1 + \frac{1}{3} \right) \\
 &= \frac{32}{3} \approx 10.6667.
 \end{aligned}$$

- (b) Evaluate the integral $\int t \cos(2t) dt$.

Solution: First we make the substitution $x = 2t$, so $dx = 2 dt$ or $dt = \frac{1}{2} dx$:

$$\int t \cos(2t) dt = \int \frac{x}{2} \cos(x) \frac{1}{2} dx = \frac{1}{4} \int x \cos(x) dx.$$

In order to integrate $\int x \cos(x) dx$, we must use integration by parts; the version I'll use looks like

$$\int u dv = uv - \int v du.$$

Here let $u = x$ and $dv = \cos(x) dx$; thus $du = dx$ and $v = \sin(x)$. Hence

$$\begin{aligned}
 \int t \cos(2t) dt &= \frac{1}{4} \int x \cos(x) dx \\
 &= \frac{1}{4} \left[x \sin(x) - \int \sin(x) dx \right] \\
 &= \frac{1}{4} x \sin(x) - \frac{1}{4} (-\cos(x)) + K \\
 &= \frac{1}{4} x \sin(x) + \frac{1}{4} \cos(x) + K.
 \end{aligned}$$

The answer should be in terms of t , so we make the substitution back

$$\begin{aligned}\int t \cos(2t) dt &= \frac{1}{4}(2t) \sin(2t) + \frac{1}{4} \cos(2t) + K \\ &= \frac{1}{2}t \sin(2t) + \frac{1}{4} \cos(2t) + K.\end{aligned}$$

This is our final answer.

(c) Evaluate the integral $\int_0^1 x\sqrt{1-x} dx$.

Solution: Perhaps the simplest way to do this integral is to make the substitution $u = 1 - x$. Then $x = 1 - u$, and the limits that were $x = 0$ to $x = 1$ are now $u = 1 - 0 = 1$ to $u = 1 - 1 = 0$:

$$\begin{aligned}\int_0^1 x\sqrt{1-x} dx &= \int_1^0 (1-u)\sqrt{u} - du \\ &= -\int_1^0 (u^{1/2} - u^{3/2}) du.\end{aligned}$$

This integral is now straightforward:

$$\begin{aligned}&= -\left[\frac{1}{3/2}u^{3/2} - \frac{1}{5/2}u^{5/2}\right]_1^0 \\ &= -\left[\frac{2}{3}(0^{3/2} - 1^{3/2}) - \frac{2}{5}(0^{5/2} - 1^{5/2})\right] \\ &= \frac{4}{15} \approx 0.2667.\end{aligned}$$

This is the value of our definite integral.

14 We are given the differential equation $y'' - 6y' + cy = 0$, where c is an unknown constant. Find c so that...

(a) ... $y = 6te^{3t}$ is a solution.

Solution: Here's one way to look at this problem. Let's try to solve this differential equation. We form the associated quadratic $r^2 - 6r + c = 0$, which has roots

$$r = \frac{+6 \pm \sqrt{36 - 4c}}{2} = \frac{6}{2} \pm \frac{1}{2}\sqrt{4(9 - c)} = 3 \pm \sqrt{9 - c}.$$

The solutions fall into one of three cases, based on the roots r of the quadratic:

$$\begin{aligned}r_1 \neq r_2 \text{ (both real)} &\implies y = Pe^{r_1 t} + Qe^{r_2 t} \\ r = r_1 = r_2 \text{ (real)} &\implies y = Pe^{rt} + Qte^{rt} \\ r = \alpha \pm i\beta &\implies y = e^{\alpha t} (P \cos(\beta t) + Q \sin(\beta t)).\end{aligned}$$

So in order to get a solution of the form $6te^{3t}$, we need to be in the second case. In particular, this means that $r = 3$ as well, so the quadratic $r^2 - 6r + c = 0$ has one repeated root: $r = 3$. This means it must factor into $(r - 3)(r - 3)$, so $r^2 - 6r + c = (r - 3)^2 = r^2 - 6r + 9$, or $r = 9$.

Looked at a slightly different way, we're saying that $3 \pm \sqrt{9 - c}$ must be 3 and 3, so $\sqrt{9 - c} = 0$, or $c = 9$.

Another approach is to simply plug in the function $y = 6te^{3t}$ and its derivatives into the original equation. We can compute (without details):

$$y = 6te^{3t} \quad y' = 6e^{3t} + 18te^{3t} \quad \text{and} \quad y'' = 36e^{3t} + 54te^{3t}.$$

We plug this in and get

$$\begin{aligned} y'' - 6y' + c &= (36e^{3t} + 54te^{3t}) - 6(6e^{3t} + 18te^{3t}) + c \cdot 6te^{3t} \\ &= 36e^{3t} + 54te^{3t} - 36e^{3t} - 108te^{3t} + 6cte^{3t} \\ &= (6c - 54)te^{3t}. \end{aligned}$$

Since this must vanish for every t , we get $6c - 54 = 0$, or $c = 9$ (as above).

(b) ... $y = 2e^t + 4e^{5t}$ is a solution.

Solution: If we proceed as in part (a), we see that the quadratic $r^2 - 6r + c$ must have roots $r_1 = 1$ and $r_2 = 5$, so it must factor into $r^2 - 6r + c = (r - 1)(r - 5) = r^2 - 6r + 5$. That is, we must have $c = 5$.

Again we could simply plug in $y = 2e^t + 4e^{5t}$ and solve for c . Since

$$y = 2e^t + 4e^{5t} \quad y' = 2e^t + 20e^{5t} \quad \text{and} \quad y'' = 2e^t + 100e^{5t},$$

we can plug this in to find

$$\begin{aligned} y'' - 6y' + c &= (2e^t + 100e^{5t}) - 6(2e^t + 20e^{5t}) + c(2e^t + 4e^{5t}) \\ &= (2 - 12 + 2c)e^t + (100 - 120 + 4c)e^{5t} \\ &= (2c - 10)e^t + (4c - 20)e^{5t}. \end{aligned}$$

Since this must vanish for every t , we get $2c - 10 = 0$ and $4c - 20 = 0$, or $c = 5$ (as above).

(c) ... $y = 2e^{3t} \cos(2t)$ is a solution.

Solution: If we again proceed as in part (a), we see that the quadratic $r^2 - 6r + c$ must have roots $r = 3 \pm 2i$, so we must have

$$r = 3 \pm \sqrt{9 - c} = 3 \pm \sqrt{-4}.$$

That is, we must have $9 - c = -4$, or $c = 13$.

Again we could simply plug in $y = 2e^{3t} \cos(2t)$ and solve for c . Since

$$y = 2e^{3t} \cos(2t) \quad y' = 6e^{3t} \cos(2t) - 4e^{3t} \sin(2t) \quad \text{and} \quad y'' = 10e^{3t} \cos(2t) - 24e^{3t} \sin(2t),$$

we can plug this in to find

$$\begin{aligned} y'' - 6y' + c &= (10e^{3t} \cos(2t) - 24e^{3t} \sin(2t)) - 6(6e^{3t} \cos(2t) - 4e^{3t} \sin(2t)) + c(2e^{3t} \cos(2t)) \\ &= (10 - 36 + 2c)e^{3t} \cos(2t) + (-24 + 24 + 0)e^{3t} \sin(2t) \\ &= (2c - 26)e^{3t} \cos(2t). \end{aligned}$$

Since this must vanish for every t , we get $2c - 26 = 0$, or $c = 13$ (as above).

- 15 The telephone company performed a survey over several years to find what percentage of the population still use rotary (dial) telephones. They collected the following data:

Year of survey (x or t)	1	2	3	4
Percent using rotary phones (y)	20	15	5	4

- (a) Use the method of linear regression (least squares) to find a formula $y = mx + b$ that describes the percentage using rotary phones in year x . (Keep 4 digits past the decimal in your calculations.)

Solution: As in Problem 10, there are two approaches to finding the best fit line $y = mx + b$. The first is the linear algebra approach, which involves looking at the system $A^TAX = A^TB$, where

$$A = \begin{bmatrix} x_1 & 1 \\ x_2 & 1 \\ \vdots & \vdots \\ x_n & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 2 & 1 \\ 3 & 1 \\ 4 & 1 \end{bmatrix}, \quad X = \begin{bmatrix} m \\ b \end{bmatrix}, \quad \text{and} \quad Y = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix} = \begin{bmatrix} 20 \\ 15 \\ 5 \\ 4 \end{bmatrix}.$$

Then the linear system $A^TAX = A^TB$ is

$$\begin{bmatrix} 1 & 2 & 3 & 4 \\ 1 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 2 & 1 \\ 3 & 1 \\ 4 & 1 \end{bmatrix} \begin{bmatrix} m \\ b \end{bmatrix} = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 1 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 20 \\ 15 \\ 5 \\ 4 \end{bmatrix}$$

or

$$\begin{bmatrix} 30 & 10 \\ 10 & 4 \end{bmatrix} \begin{bmatrix} m \\ b \end{bmatrix} = \begin{bmatrix} 81 \\ 44 \end{bmatrix}. \tag{1}$$

On the other hand, we could also use the partial derivatives approach, from which we've learned that the linear system to solve in order to find m and b is

$$\begin{bmatrix} \sum_{i=1}^n x_i^2 & \sum_{i=1}^n x_i \\ \sum_{i=1}^n x_i & n \end{bmatrix} \begin{bmatrix} m \\ b \end{bmatrix} = \begin{bmatrix} \sum_{i=1}^n x_i y_i \\ \sum_{i=1}^n y_i \end{bmatrix}.$$

Let's compute with the data:

i	x_i	y_i	x_i^2	$x_i y_i$
1	1	20	1	20
2	2	15	4	30
3	3	5	9	15
4	4	4	16	16
Sum:	10	44	30	81

We can then see the linear system is precisely that shown in equation (1). We solve this system to see that

$$m = \frac{-116}{20} = -5.8 \quad \text{and} \quad b = 25.5.$$

Thus the least-squares line is $y = -5.8x + 25.5$.

- (b) Use the model equation you have derived in (a) to predict the year when the percentage using rotary phones is 0.

Solution: This asks the question: what is x when $y = 0$? We solve $0 = -5.8x + 25.5$ for x to get

$$x = \frac{25.5}{5.8} \approx 4.3966 \text{ years.}$$

16 Suppose the average life span of a new cellular phone is 50 months, and that these life spans are exponentially distributed.

(a) Find the probability density function $f(x)$ for this distribution.

Solution: The probability density function $f(x)$ for an exponential distribution is $f(x) = \lambda e^{-\lambda x}$ ($0 \leq x < \infty$), where $E(X) = \frac{1}{\lambda}$. Since we are told $E(X) = 50$, then we have $\frac{1}{\lambda} = 50$ or $\lambda = \frac{1}{50} = 0.02$. Thus the probability density function is $f(x) = \frac{1}{50}e^{-x/50}$.

(b) Find the probability that a cellular phone will last more than 2 years (24 months).

Solution: This question asks for $\Pr(X \geq 24)$ (since we used months in part (a), we have to use months here as well). This is given by

$$\begin{aligned} \Pr(X \geq 24) &= \int_{24}^{\infty} f(x) dx \\ &= \int_{24}^{\infty} \frac{1}{50} e^{-x/50} dx \\ &= \lim_{b \rightarrow \infty} \int_{24}^b \frac{1}{50} e^{-x/50} dx \\ &= \lim_{b \rightarrow \infty} -e^{-x/50} \Big|_{24}^b \\ &= \lim_{b \rightarrow \infty} \left[-e^{-b/50} - (-e^{-24/50}) \right] \\ &= e^{-24/50} \approx 0.6188. \end{aligned}$$

Thus the probability is roughly 61.88%.

(c) Find the probability that a cellular phone will fail within the first 2 years (24 months).

Solution: Here we could perform the same sort of integral as in part (b), or we could simply notice that $\Pr(0 \leq X \leq 24) = 1 - \Pr(X \geq 24)$. Since we have already calculated $\Pr(X \geq 24)$, we can simply use this: $\Pr(0 \leq X \leq 24) = 1 - \Pr(X \geq 24) = 1 - e^{-24/50} \approx 0.3812$. Thus there is a 38.12% chance that a cellular phone (of this type) will fail within the first 24 months.

17 (a) For what value of k will $f(x) = kx^2(1-x)$, $0 \leq x \leq 1$, be a probability density function?

Solution: This is a probability density function when $\int_a^b f(x) dx = 1$. We calculate:

$$\begin{aligned} 1 &= \int_0^1 kx^2(1-x) dx \\ &= k \int_0^1 (x^2 - x^3) dx \\ &= k \left[\frac{1}{3}x^3 - \frac{1}{4}x^4 \right]_0^1 \\ &= k \left[\left(\frac{1}{3}1^3 - \frac{1}{4}1^4 \right) - \left(\frac{1}{3}0^3 - \frac{1}{4}0^4 \right) \right] \\ &= k \left(\frac{1}{3} - \frac{1}{4} \right) \\ &= \frac{1}{12}k. \end{aligned}$$

Thus $k = 12$.

- (b) Find $\Pr(X \leq 1/2)$ for a random variable X with probability density function $f(x)$ (as in part (a)).

Solution: Recall that

$$\Pr(X \leq 1/2) = \int_a^{1/2} f(x) dx.$$

Here $a = 0$ and $f(x) = 12x^2(1-x) = 12x^2 - 12x^3$, as computed in part (a). One anti-derivative of $f(x)$ is $F(x) = 4x^3 - 3x^4$. This means that

$$\begin{aligned} \Pr(X \leq 1/2) &= \int_0^{1/2} f(x) dx \\ &= (4x^3 - 3x^4) \Big|_0^{1/2} \\ &= \left[4 \left(\frac{1}{2} \right)^3 - 3 \left(\frac{1}{2} \right)^4 \right] - [4(0)^3 - 3(0)^4] \\ &= \frac{5}{16} = 0.3125. \end{aligned}$$

Thus the probability that X is less than (or equal to) $1/2$ is 31.25%.

- (c) Find $E(X)$, the expected value of a random variable X with probability density function $f(x)$ (as in part (a)).

Solution: The mean (or expected value) is given by

$$E(X) = \int_a^b x f(x) dx.$$

Here $a = 0$, $b = 1$, and $f(x) = 12x^2(1-x)$. Thus we have

$$\begin{aligned} E(X) &= \int_0^1 x \cdot 12x^2(1-x) dx \\ &= 12 \int_0^1 (x^3 - x^4) dx \\ &= 12 \left[\frac{1}{4}x^4 - \frac{1}{5}x^5 \right]_0^1 \\ &= 12 \left[\left(\frac{1}{4}1^4 - \frac{1}{5}1^5 \right) - \left(\frac{1}{4}0^4 - \frac{1}{5}0^5 \right) \right] \\ &= \frac{3}{5} = 0.6. \end{aligned}$$

Thus $E(X) = 0.6$.

- 18 (a) Find the solution to the initial value problem

$$\begin{aligned} y' &= \frac{t^3}{y^2} \\ y(0) &= 1. \end{aligned}$$

Solution: The given differential equation is separable:

$$\begin{aligned} \frac{dy}{dt} &= \frac{t^3}{y^2} \\ y^2 dy &= t^3 dt. \end{aligned}$$

Now we integrate both sides and solve for y :

$$\begin{aligned}\frac{1}{3}y^3 &= \frac{1}{4}t^4 + K \\ y^3 &= \frac{3}{4}t^4 + k \quad (k = 3K) \\ y &= \sqrt[3]{\frac{3}{4}t^4 + k}.\end{aligned}$$

Now we solve for the unknown constant k by plugging in the initial condition $y(0) = 1$:

$$1 = \sqrt[3]{\frac{3}{4}(0)^4 + k} = \sqrt[3]{k}.$$

Cubing both sides gives us $k = 1^3 = 1$, so $y = \sqrt[3]{\frac{3}{4}t^4 + 1}$ or, written another way $y = \left(\frac{3}{4}t^4 + 1\right)^{1/3}$.

(b) Find the general solution to the differential equation

$$y' + \frac{2}{x}y = \frac{\sin(x^2)}{x} \quad (x > 0).$$

Solution: This equation is a first-order linear differential equation; that is, it is of the form

$$y' + P(x)y = G(x),$$

with $P(x) = \frac{2}{x}$ and $G(x) = \frac{\sin(x^2)}{x}$. We solve these using the integrating factor $h(x) = e^{\int P(x) dx}$. Integrating, we get $\int P(x) dx = 2 \ln(x)$, so $h(x) = e^{2 \ln(x)} = e^{\ln(x^2)} = x^2$. Thus when we multiply the equation by x^2 , we get

$$\begin{aligned}x^2y' + x^2\frac{2}{x}y &= x^2 \cdot \frac{\sin(x^2)}{x} \\ x^2y' + 2xy &= x \sin(x^2) \\ (x^2y)' &= x \sin(x^2).\end{aligned}$$

Now, integrating both sides, we get

$$x^2y = -\frac{1}{2} \cos(x^2) + K,$$

or

$$y = -\frac{1}{2x^2} \cos(x^2) + \frac{K}{x^2}.$$

This is the general solution.