7 Let $A = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 1 \\ 0 & 1 & 2 \end{bmatrix}$.

(a) Find A^{-1} by hand (show your calculations).

Solution: The simplest way to do this is to row reduce $[A \mid I\,]$ into $[\,I \mid A^{-1}]$, then read the inverse matrix from this form. We perform this calculation:

$$[A \mid I\,] = \left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 & 1 & 0 \\ 0 & 1 & 2 & 0 & 0 & 1 \end{array}\right]$$

$$\longrightarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & -1 & 1 & 0 \\ 0 & 1 & 2 & 0 & 0 & 1 \end{array}\right] \qquad (R_2 = r_2 - r_1)$$

$$\longrightarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & -1 & 1 & 0 \\ 0 & 0 & 2 & 1 & -1 & 1 \end{array}\right] \qquad (R_3 = r_3 - r_2)$$

$$\longrightarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & -1 & 1 & 0 \\ 0 & 0 & 1 & 0.5 & -0.5 & 0.5 \end{array}\right] \qquad (R_3 = \frac{1}{2} r_3)$$

$$\longrightarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 0.5 & 0.5 & -0.5 \\ 0 & 1 & 0 & -1 & 1 & 0 \\ 0 & 0 & 1 & 0.5 & -0.5 & 0.5 \end{array}\right] \qquad (R_1 = r_1 - r_3)$$

$$= [\,I \mid A^{-1}].$$

Thus

$$A^{-1} = \begin{bmatrix} 0.5 & 0.5 & -0.5 \\ -1 & 1 & 0 \\ 0.5 & -0.5 & 0.5 \end{bmatrix}.$$

(b) Use part (a) to solve the following system:

$$\begin{aligned} x \quad\;\; + \; z &= \;\;1 \\ x + y + \; z &= -1 \\ y + 2z &= \;\;0 \end{aligned}$$

Solution: This system may be written as $AX = B$, or

$$\begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 1 \\ 0 & 1 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}.$$

Thus one way to solve this is by multiplying by A^{-1}: $X = A^{-1}B$. We do this calculation:

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0.5 & 0.5 & -0.5 \\ -1 & 1 & 0 \\ 0.5 & -0.5 & 0.5 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0.5 - 0.5 + 0 \\ -1 - 1 + 0 \\ 0.5 + 0.5 + 0 \end{bmatrix} = \begin{bmatrix} 0 \\ -2 \\ 1 \end{bmatrix}.$$

One can check directly that $(x, y, z) = (0, -2, 1)$ satisfy the given linear system.

(c) If one changes the number 2 in matrix A to some other number, say k, one can make the new matrix have no inverse. Find k.

Solution: Let's try to compute the inverse of

$$A = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 1 \\ 0 & 1 & k \end{bmatrix}$$

as we did in part (a): by using elimination to transform $[A \mid I\,]$ into $[\,I \mid A^{-1}]$. The first two steps are as in part (a):

$$[A \mid I\,] = \left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 & 1 & 0 \\ 0 & 1 & k & 0 & 0 & 1 \end{array}\right]$$

$$\longrightarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & -1 & 1 & 0 \\ 0 & 0 & k & 1 & -1 & 1 \end{array}\right] \qquad (R_2 = r_2 - r_1, R_3 = r_3 - r_2).$$

Now we'll be able to proceed as in part (a) provided we can divide by k. That is, we will only fail to find A^{-1} if we cannot divide by k, which only happens if $k = 0$. Thus A is invertible provided $k \neq 0$.

8 Let $A = \begin{bmatrix} 1 & 3 \\ 2 & 6 \end{bmatrix}$ and $B = \begin{bmatrix} 5 & 3 \\ 4 & 6 \end{bmatrix}$.

(a) Does $(A+B)^{-1}$ exist? Verify your answer with an appropriate calculation.

Solution: The simplest way, perhaps, to do this is to try to compute $(A+B)^{-1}$ and see what happens. We compute $A+B$ first:

$$A + B = \begin{bmatrix} 1 & 3 \\ 2 & 6 \end{bmatrix} + \begin{bmatrix} 5 & 3 \\ 4 & 6 \end{bmatrix} = \begin{bmatrix} 1+5 & 3+3 \\ 2+4 & 6+6 \end{bmatrix} = \begin{bmatrix} 6 & 6 \\ 6 & 12 \end{bmatrix}.$$

This matrix has inverse

$$(A+B)^{-1} = \frac{1}{6(12) - 6(6)} \begin{bmatrix} 12 & -6 \\ -6 & 6 \end{bmatrix} = \frac{1}{36} \begin{bmatrix} 12 & -6 \\ -6 & 6 \end{bmatrix} = \begin{bmatrix} 1/3 & -1/6 \\ -1/6 & 1/6 \end{bmatrix}.$$

Thus $(A+B)^{-1}$ exists.

(b) Does $(A-B)^{-1}$ exist? Verify your answer with an appropriate calculation.

Solution: This is extremely similar to part (a). We compute $A - B$ first:

$$A - B = \begin{bmatrix} 1 & 3 \\ 2 & 6 \end{bmatrix} - \begin{bmatrix} 5 & 3 \\ 4 & 6 \end{bmatrix} = \begin{bmatrix} 1-5 & 3-3 \\ 2-4 & 6-6 \end{bmatrix} = \begin{bmatrix} -4 & 0 \\ -2 & 0 \end{bmatrix}.$$

Now when we try to compute $(A-B)^{-1}$, we run into trouble:

$$(A-B)^{-1} = \frac{1}{-4(0) - (-2)(0)} \begin{bmatrix} 0 & 0 \\ 2 & -4 \end{bmatrix} = \frac{1}{0} \begin{bmatrix} 0 & 0 \\ 2 & -4 \end{bmatrix}.$$

Since this involves division by zero, $(A-B)^{-1}$ does not exist.

(c) The problem $AX - BX = Y$ is equivalent to $(A-B)X = Y$. Let $Y = \begin{bmatrix} -10 \\ -5 \end{bmatrix}$.

Say Gaussian elimination is applied to this problem. The augmented matrix $[(A-B) \mid Y]$ is reduced by forward elimination to the matrix $\left[\begin{array}{cc|c} 1 & 0 & 5/2 \\ 0 & 0 & 0 \end{array}\right]$.

Write an expression for the solution(s) of this system. If no solutions exist, then state that along with something to back it up.

Solution: This last augmented matrix is equivalent to the linear system

$$1x + 0y = \frac{5}{2}$$
$$0x + 0y = 0.$$

This implies that $x = 5/2$, but y can be anything. That is, $X = \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 5/2 \\ y \end{bmatrix}$ is a solution to $AX - BX = Y$ for *any* value of y. These are the solutions to the system.

(d) Given A and B above and your previous results, can you say if the system in (c) has a unique solution? Explain.

Solution: There are an infinite number of solutions to the system in part (c). The solutions to the original system are precisely the solutions to the reduced system; hence there are infinitely many solutions $X = \begin{bmatrix} 5/2 \\ y \end{bmatrix}$ to the original system as well.

9 Let

$$A = \begin{bmatrix} 1 & 1 & -1 \\ 2 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix}.$$

Does A^{-1} exist? Explain how you know. If it exists, find A^{-1}.

Solution: We can "explain how we know" that A^{-1} exists only by finding the inverse. (There are ways to tell if a matrix is invertible that we didn't cover. The first part of this question was originally intended to test one of them. Students were supposed to compute the *determinant* of A; since this determinant is not zero, the matrix A is invertible. We didn't cover this, so we'll just proceed with trying to compute A^{-1}.) We compute A^{-1} in the usual way: we row reduce $[A \mid I\,]$ into $[\,I \mid A^{-1}]$, then read the inverse matrix from this form:

$$[A \mid I\,] = \left[\begin{array}{ccc|ccc} 1 & 1 & -1 & 1 & 0 & 0 \\ 2 & 1 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 & 1 \end{array}\right]$$

$$\longrightarrow \left[\begin{array}{ccc|ccc} 1 & 1 & -1 & 1 & 0 & 0 \\ 0 & -1 & 3 & -2 & 1 & 0 \\ 0 & -1 & 2 & -1 & 0 & 1 \end{array}\right] \qquad (R_2 = r_2 - 2r_1, R_3 = r_3 - r_1)$$

$$\longrightarrow \left[\begin{array}{ccc|ccc} 1 & 1 & -1 & 1 & 0 & 0 \\ 0 & 1 & -3 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 & 0 & 1 \end{array}\right] \qquad (R_2 = -r_2)$$

$$\longrightarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 2 & -1 & 1 & 0 \\ 0 & 1 & -3 & 2 & -1 & 0 \\ 0 & 0 & -1 & 1 & -1 & 1 \end{array}\right] \qquad (R_1 = r_1 - r_2, R_3 = r_3 + r_2)$$

$$\longrightarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 2 & -1 & 1 & 0 \\ 0 & 1 & -3 & 2 & -1 & 0 \\ 0 & 0 & 1 & -1 & 1 & -1 \end{array}\right] \qquad (R_3 = -r_3)$$

$$\longrightarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & -1 & 2 \\ 0 & 1 & 0 & -1 & 2 & -3 \\ 0 & 0 & 1 & -1 & 1 & -1 \end{array}\right] \qquad (R_1 = r_1 - 2r_3, R_2 = r_2 + 3r_3)$$

$$= [\,I \mid A^{-1}].$$

Thus A^{-1} exists, and

$$A^{-1} = \begin{bmatrix} 1 & -1 & 2 \\ -1 & 2 & -3 \\ -1 & 1 & -1 \end{bmatrix}.$$

10 Data giving the number of hours a student has studied compared to his or her performance on an exam (of maximum 100 points) are given in the following table:

Hours Studied	0	3	6	9
Exam Score	50	62	81	92

(a) Find a least squares line that best fits the given data.

Solution: We will let x_i be the Hours Studied (the independent variable) and y_i be the Exam Score (the dependent variable). There are then two approaches to finding the best fit line $y = mx + b$. The first is the linear algebra approach, which involves looking at the system $A^T AX = A^T B$, where

$$A = \begin{bmatrix} x_1 & 1 \\ x_2 & 1 \\ \vdots & \vdots \\ x_n & 1 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 3 & 1 \\ 6 & 1 \\ 9 & 1 \end{bmatrix}, \qquad X = \begin{bmatrix} m \\ b \end{bmatrix}, \qquad \text{and} \qquad Y = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix} = \begin{bmatrix} 50 \\ 62 \\ 81 \\ 92 \end{bmatrix}.$$

Then the linear system $A^T AX = A^T B$ is

$$\begin{bmatrix} 126 & 18 \\ 18 & 4 \end{bmatrix} \begin{bmatrix} m \\ b \end{bmatrix} = \begin{bmatrix} 1500 \\ 285 \end{bmatrix}. \tag{1}$$

On the other hand, we could also use the partial derivatives approach, from which we've learned that the linear system to solve in order to find m and b is

$$\begin{bmatrix} \sum_{i=1}^{n} x_i^2 & \sum_{i=1}^{n} x_i \\ \sum_{i=1}^{n} x_i & n \end{bmatrix} \begin{bmatrix} m \\ b \end{bmatrix} = \begin{bmatrix} \sum_{i=1}^{n} x_i y_i \\ \sum_{i=1}^{n} y_i \end{bmatrix}.$$

If we compute with the data:

i	x_i	y_i	x_i^2	$x_i y_i$
1	0	50	0	0
2	3	62	9	186
3	6	81	36	486
4	9	92	81	828
Sum:	18	285	126	1500

We can then see the linear system is precisely that shown in equation (1). We solve this system to see that

$$m = \frac{29}{6} \approx 4.8333 \qquad \text{and} \qquad b = 49.5.$$

Thus the least squares line is $y = \frac{29}{6}x + 49.5$.

(b) What prediction would this line make for a student who studied 10 hours?

Solution: The question asks: what is y (the exam score) when a student has studied $x = 10$ hours. Our best guess is $y = m(10) + b \approx 97.83$, or about a 97.8.

11 Suppose an assembly line produces a small mechanical toy, and about 1% of the toys are defective. A quality control worker chooses a toy at random, inspects it, and then repeats this process until a defective toy is found.

(a) Find the probability that at most 2 toys pass the test before a defective toy is found.

Solution: Let X be the random variable equal to the number of toys that pass inspection before the first failure. How can we compute $\Pr(X = k)$? This is the probability that k toys pass the inspection, then the next one (the $k + 1$st) fails. Since each toy passes with probability $p = 0.99$ (and fails with probability $q = 1 - p = 0.01$, this probability is

$$\Pr(X = k) = p \cdot p \cdots p \cdot q = p^k q.$$

Thus

$$\begin{aligned}\Pr(X \leq 2) &= \Pr(X = 0) + \Pr(X = 1) + \Pr(X = 2)\\ &= q + pq + p^2 q\\ &= 0.01 + 0.99(0.01) + (0.99)^2(0.01)\\ &= 0.029701.\end{aligned}$$

Thus there is roughly a 2.97% chance that at most 2 toys pass the test before a defective toy is found.

(b) What is the probability that at least 3 toys will pass the test before the first defective toy is found?

Solution: The probability that at least 3 toys will pass the test before the first defective toy is found is simply

$$\Pr(X \geq 3) = 1 - \Pr(X \leq 2).$$

Since we've computed $\Pr(X \leq 2)$ in part (a), this is just subtraction:

$$\Pr(X \geq 3) = 1 - \Pr(X \leq 2) = 1 - 0.029701 = 0.970299.$$

That is, there is a roughly 97.03% chance that at least 3 toys will pass the test before a defective toy is found.

12 The SACT is a new test for college-bound high school students. It is designed so that scores are normally distributed with a mean of 400 and a standard deviation of 100.

(a) Find the probability that a randomly selected high school student scores between 300 and 450 on the SACT.

Solution: This asks for $\Pr(300 \leq X \leq 450)$, where X is a normal random variable with mean $\mu = 400$ and standard deviation $\sigma = 100$. We calculate this in the usual way, by changing it into a question about the standard normal variable Z:

$$\begin{aligned}\Pr(300 \leq X \leq 450) &= \Pr\left(\frac{300-\mu}{\sigma} \leq Z \leq \frac{450-\mu}{\sigma}\right)\\ &= \Pr\left(\frac{300-400}{100} \leq Z \leq \frac{450-400}{100}\right)\\ &= \Pr\left(-1 \leq Z \leq 0.5\right).\end{aligned}$$

Now we break this into two probabilities and exploit the symmetry of the standard normal distribution:

$$\begin{aligned}\Pr(300 \leq X \leq 450) &= \Pr\left(-1 \leq Z \leq 0.5\right)\\ &= \Pr\left(-1 \leq Z \leq 0\right) + \Pr\left(0 \leq Z \leq 0.5\right)\\ &= \Pr\left(0 \leq Z \leq 1\right) + \Pr\left(0 \leq Z \leq 0.5\right) \qquad \text{by symmetry.}\end{aligned}$$

Now we can find these probabilities on our standard normal distribution table: $\Pr(0 \leq Z \leq 1) = 0.3413$ and $\Pr(0 \leq Z \leq 0.5) = 0.1915$. Thus

$$\Pr(300 \leq X \leq 450) = 0.3413 + 0.1915 = 0.5328.$$

That is, the probability that a randomly selected high school student scores between 300 and 450 on the SACT is about 53.28%.

(b) Find the probability that a randomly selected high school student scores over 650 on the SACT.

Solution: This asks for $\Pr(X \geq 650)$, where X is a normal random variable with mean $\mu = 400$ and standard deviation $\sigma = 100$. We calculate this in the usual way, by changing it into a question about the standard normal variable Z:

$$\Pr(X \geq 650) = \Pr\left(Z \geq \frac{650-\mu}{\sigma}\right) = \Pr\left(Z \geq \frac{650-400}{100}\right) = \Pr\left(Z \geq 2.5\right).$$

Now we use the symmetry of the standard normal distribution. Since the distribution is symmetric about $\mu = 0$, we know that $\Pr(Z \geq 0) = 0.5$ (and $\Pr(Z \leq 0) = 0.5$, but we won't use this). Thus

$$\Pr\left(Z \geq 2.5\right) = \Pr\left(Z \geq 0\right) - \Pr\left(0 \leq Z \leq 2.5\right).$$

These two probabilities on the right we can find: the first is $\Pr\left(Z \geq 0\right) = 0.5$ and the second is (from the table) $\Pr\left(0 \leq Z \geq 2.5\right) = 0.4938$. Thus

$$\Pr\left(Z \geq 2.5\right) = \Pr\left(Z \geq 0\right) - \Pr\left(0 \leq Z \leq 2.5\right) = 0.5 - 0.4938 = 0.0062.$$

That is, the probability that a randomly selected high school student scores over 650 on the SACT is about 0.62%.

(c) It turns out that there was a mistake in the computations. The mean of the SACT is 400, but the standard deviation is *not* 100. One school is told that a 520 is the cut-off for scores in the top 5%. Assuming this is correct, what is the standard deviation?

Solution: Now we're told that $\Pr(X \geq 520) = 0.05$ for a normal distribution X with mean $\mu = 400$ and we're asked to find the standard deviation σ. Again we turn to the standard normal distribution Z. We'd like to find the value of z for which $\Pr(Z \geq z) = 0.05$. As in part (b), this is the value for which $\Pr(0 \leq Z \leq z) = 0.5 - 0.05 = 0.45$.

The table tells us that $\Pr(0 \leq Z \leq 1.64) = 0.4495$ and $\Pr(0 \leq Z \leq 1.65) = 0.4505$. We can then interpolate and say that $\Pr(0 \leq Z \leq 1.645) \approx 0.4500$ (since 1.645 is half-way between 1.64 and 1.65, just 0.45 is mid-way between 0.4495 and 0.4505). That is, we get $z \approx 1.645$ is the value for which $\Pr(Z \geq z) = 0.05$.

Finally, we relate the two distributions X and Z. In the usual way,

$$0.05 = \Pr(X \geq 520) = \Pr\left(Z \geq \frac{520-\mu}{\sigma}\right) = \Pr\left(Z \geq \frac{520-400}{\sigma}\right) = \Pr\left(Z \geq \frac{120}{\sigma}\right).$$

We found in the previous paragraph that $\Pr(Z \geq 1.645) = 0.05$, so $\frac{120}{\sigma} = 1.645$. We solve for the standard deviation to find that $\sigma \approx \frac{120}{1.645} \approx 72.95$. That is, the standard deviation of distribution of SACT scores is approximately 73.

One final comment: in the second paragraph above, we interpolated and used $z = 1.645$ rather than $z = 1.64$ or $z = 1.65$. The standard deviation would have ranged between $\sigma \approx 72.73$ (for $z = 1.65$) and $\sigma \approx 73.17$ (for $z = 1.64$), so an approximation of $\sigma \approx 73$ works for all three options.