1 Evaluate the following integrals:

(a) $\int \frac{x}{2} \sin(x/2)\ dx$

Solution: This integral is slightly simpler if we make the substitution $t = x/2$, so $dt = \frac{1}{2}\ dx$ or $dx = 2\ dt$. Then the integral becomes

$$\int \frac{x}{2} \sin\left(\frac{x}{2}\right)\ dx = \int t \sin(t)\ 2\ dt = 2 \int t \sin(t)\ dt.$$

Now this integral requires an integration by parts. We'll use the integration by parts formula in its form

$$\int u\ dv = uv - \int v\ du.$$

If we let $u = t$ and $dv = \sin(t)\ dt$, then $du = dt$ and $v = -\cos(t)$. Hence

$$\begin{aligned}\int \frac{x}{2} \sin\left(\frac{x}{2}\right)\ dx &= 2 \int t \sin(t)\ dt \\ &= 2\left[t\left(-\cos(t)\right) - \int \left(-\cos(t)\right)\ dt\right] \\ &= -2t\cos(t) + 2\int \cos(t)\ dt \\ &= -2t\cos(t) + 2\sin(t) + C.\end{aligned}$$

Now we return to x via $t = x/2$ to get our answer:

$$\int \frac{x}{2} \sin\frac{x}{2}\ dx = -x\cos(x/2) + 2\sin(x/2) + C.$$

(b) $\int_{-1}^{1} x^2\sqrt{1 - x^3}\ dx$

Solution: For this integral we will make the substitution $u = 1 - x^3$. (We've chosen this substitution so we'll end up with a $\sqrt{u}$ in the integral.) From this we get that $du = -3x^2\ dx$, or $x^2\ dx = -\frac{1}{3}\ du$. (Also notice that when $x = -1$, $u = 1 - (-1)^3 = 2$; when $x = 1$, $u = 1 - 1^3 = 0$.) Hence

$$\begin{aligned}\int_{-1}^{1} x^2\sqrt{1 - x^3}\ dx &= \int_{-1}^{1} \sqrt{1 - x^3} \cdot x^2\ dx \\ &= \int_{x=-1}^{x=1} \sqrt{u} \cdot -\frac{1}{3}\ du \\ &= -\frac{1}{3}\int_{u=2}^{u=0} \sqrt{u}\ du \\ &= -\frac{1}{3}\int_{2}^{0} u^{1/2}\ du.\end{aligned}$$

This is now a simple integral:

$$\begin{aligned}\int_{-1}^{1} x^2\sqrt{1 - x^3}\ dx &= -\frac{1}{3}\frac{1}{3/2}u^{3/2}\Big|_2^0 \\ &= -\frac{2}{9}\left(0^{3/2} - 2^{3/2}\right) \\ &= \frac{2}{9}2\sqrt{2} \\ &= \frac{4\sqrt{2}}{9} \approx 0.628539.\end{aligned}$$

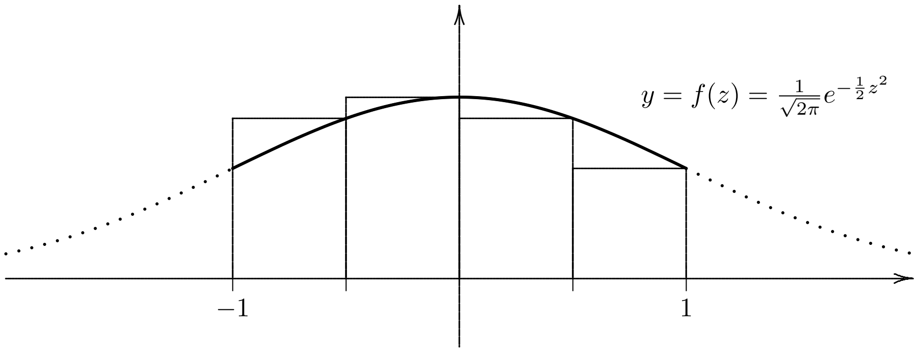
2 The standard normal curve may be written $f(z) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}z^2}$. The area under $f(z)$ from 0 to z may be written $A(z) = \int_0^z f(z)dz$.

For a random variable Z having a standard normal density, the probability $\Pr(-1 \leq Z \leq 1) \approx 0.68$ (this is the probability that Z is within one standard deviation of the mean). This value is the area under $f(z)$ from -1 to 1.

Estimate this area (or probability) in two ways:

(a) Divide the interval $[-1, 1]$ into $n = 4$ subintervals and use the right-hand endpoint of each subinterval as u.

Solution: The area we're interested in estimating is the area under the solid curve, below. We will approximate this area by the area of the four rectangles in the picture.



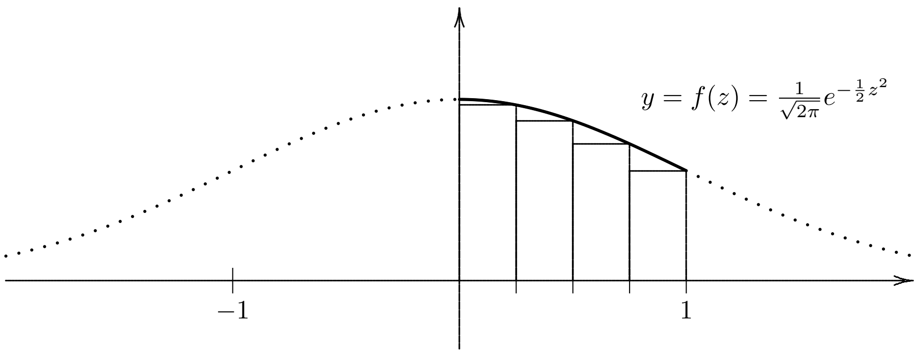
These rectangles all have width ½; their heights are (left to right) $f(-0.5)$, $f(0)$, $f(0.5)$, and $f(1)$. Thus their total area is

$$\begin{aligned}\text{Area of four rectangles} &= \frac{1}{2}f(-0.5) + \frac{1}{2}f(0) + \frac{1}{2}f(0.5) + \frac{1}{2}f(1) \\ &= \frac{1}{2}\left(\frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}(-0.5)^2} + \frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}(0)^2} + \frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}(0.5)^2} + \frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}(1)^2}\right) \\ &\approx \frac{1}{2}(0.35207 + 0.39894 + 0.35207 + 0.24197) \\ &\approx 0.6725.\end{aligned}$$

Thus the area under the curve from $z = -1$ to $z = 1$ (or the probability $\Pr(-1 \leq Z \leq 1)$) is about 0.6725.

(b) Notice that $f(z)$ is symmetric (that is, $f(z) = f(-z)$; draw the picture for yourself), so the area from $z = -1$ to $z = 0$ is the same as the area from $z = 0$ to $z = 1$. So estimate the area from 0 to 1 (using $n = 4$ subintervals and the right-hand endpoint, as in part (a)), then double that value.

Solution: Now the picture looks like this:



Now the area of the four rectangles is

$$\begin{aligned}\text{Area of four rectangles} &= \frac{1}{4}f(0.25) + \frac{1}{4}f(0.5) + \frac{1}{4}f(0.75) + \frac{1}{4}f(1)\\ &= \frac{1}{4}\left(\frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}(0.25)^2} + \frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}(0.5)^2} + \frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}(0.75)^2} + \frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}(1)^2}\right)\\ &\approx \frac{1}{4}(0.38667 + 0.35207 + 0.30114 + 0.24197)\\ &\approx 0.32046.\end{aligned}$$

Now we double this value to get that the area under the curve from $z = -1$ to $z = 1$ (or the probability $\Pr(-1 \leq Z \leq 1)$) is about 0.6409.

What's gone wrong? In part (a) we had only four subintervals, but we're significantly closer to the true answer (0.68) than in part (b), where we have effectively eight subintervals. But in part (a), the rectangles sometimes included too much area, and sometimes too little. Thus the estimate was close. In part (b), our rectangles always contain too little area, and therefore our estimate is low.

3 Solve the following initial value problem: $y' = -y\ \sin(\pi t)$, $y(\frac{1}{2}) = -\frac{1}{2}$

Solution: The equation $y' = -y\sin(\pi t)$ is both first-order linear and separable. We will, as usual, solve the equation both ways. First, as a separable equation: we write y' as $\frac{dy}{dt}$ and re-write this equation as

$$\frac{dy}{y} = -\sin(\pi t)\ dt.$$

Now integrate both sides:

$$\begin{aligned}\int \frac{dy}{y} &= -\int \sin(\pi t)\ dt\\ \ln(|y|) &= -\int \sin(u)\ \frac{1}{\pi}du \qquad (u = \pi t,\quad du = \pi\ \ dt,\quad dt = \frac{1}{\pi}\ du)\\ &= -\frac{1}{\pi}\int \sin(u)\ du\\ &= -\frac{1}{\pi}(-\cos(u)) + K\\ &= \frac{1}{\pi}\cos(\pi t) + K.\end{aligned}$$

Exponentiate to get $y = e^{\cos(\pi t)/\pi + K} = Ce^{\cos(\pi t)/\pi}$. Now we can find C by plugging in the initial value: $y = -1/2$ when $t = 1/2$:

$$-\frac{1}{2} = Ce^{\cos(pi/2)/\pi} = Ce^{0/\pi} = C,$$

or $y = -\frac{1}{2}e^{\cos(\pi t)/\pi}$.

We can also solve this differential equation by thinking of it as a first-order linear differential equation. We write the equation in the standard form

$$y' + \sin(\pi t)y = 0,$$

so $P(t) = \sin(\pi t)$ and $G(t) = 0$. The integrating factor is therefore

$$h(t) = e^{\int P(t)\ dt} = e^{\int \sin(\pi t)\ dt} = e^{-\frac{1}{\pi}\cos(\pi t)} = e^{-\cos(\pi t)/\pi}.$$

(We've done this integral by making the substitution $u = \pi t$, so $du = \pi\ dt$ and $dt = \frac{1}{\pi}\ du$.) We multiply through by the integrating factor to get

$$e^{-\cos(\pi t)/\pi}(y' + \sin(\pi t)y) = 0 \qquad \text{or} \qquad \left(e^{-\cos(\pi t)/\pi}y\right)' = 0.$$

Integrating, we get $e^{-\cos(\pi t)/\pi}y = K$, or $y = Ke^{\cos(\pi t)/\pi}$. Now we continue as in the previous part to find $K = -1/2$, or $y = -\frac{1}{2}e^{\cos(\pi t)/\pi}$, as before.

4 (a) For what value k is $f(x) = k\cos\left(\frac{\pi}{2}x\right)$ a probability function on the interval $-1 \le x \le 1$?

Solution: The constant k may be found by checking that $\int_{-1}^{1} f(x)\,dx = 1$. This is simply a computation:

$$\begin{aligned}
1 &= \int_{-1}^{1} k\cos(\frac{\pi}{2}x)\,dx \\
&= k\int_{u=-\pi/2}^{\pi/2} \cos(u)\,\frac{2}{\pi}\,du \qquad u = \frac{\pi}{2}x, \text{ so } du = \frac{\pi}{2}\,dx \text{ or } dx = \frac{2}{\pi}\,du \\
&= \frac{2k}{\pi}\int_{u=-\pi/2}^{u=\pi/2} \cos(u)\,du \\
&= \frac{2k}{\pi}\sin(u)\Big|_{u=-\pi/2}^{u=\pi/2} \\
&= \frac{2k}{\pi}[\sin(\pi/2) - \sin(-\pi/2)] \\
&= \frac{2k}{\pi}[1-(-1)] \\
&= \frac{4k}{\pi}.
\end{aligned}$$

Thus $k = \frac{\pi}{4}$.

(b) If X is a continuous random variable whose probabilities follow this density and distribution, find $\Pr(X > \frac{1}{2})$.

Solution: This probability may be computed as

$$\Pr(X > \frac{1}{2}) = \Pr(\frac{1}{2} < X \le 1) = \int_{1/2}^{1} f(x)\,dx$$

(since $f(x)$ is defined only on the interval $-1 \le x \le 1$). We've already computed the anti-derivative in part (a):

$$\int f(x)\,dx = \int k\cos(\frac{\pi}{2}x)\,dx = \frac{2k}{\pi}\sin(\frac{\pi}{2}x) + K = \frac{1}{2}\sin(\frac{\pi}{2}x) + K$$

(since $k = \pi/4$ and, in the computation in part (a), $u = \frac{\pi}{2}x$). Since we're dealing with a definite integral, the constant K is irrelevant:

$$\begin{aligned}
\Pr(X > \frac{1}{2}) &= \int_{1/2}^{1} f(x)\,dx = \frac{1}{2}\sin(\frac{\pi}{2}x)\Big|_{1/2}^{1} \\
&= \frac{1}{2}\sin(\frac{\pi}{2}) - \frac{1}{2}\sin(\frac{\pi}{4}) \\
&= \frac{1}{2}\cdot 1 - \frac{1}{2}\cdot\frac{\sqrt{2}}{2} \\
&= \frac{1}{2}\left(1 - \frac{\sqrt{2}}{2}\right) \approx 0.14647.
\end{aligned}$$

Thus the probability that X is over 1/2 is roughly 14.65%.

5 An experiment is performed to study the evaporation of alcohol. The following data are collected:

Time after start of experiment (x or t in hours)	0	2	4	6	8	10
Amount of alcohol remaining (y in liters)	1.00	0.85	0.72	0.59	0.45	0.31

(a) Use the method of linear regression (least squares) to find a formula $y = mx + b$ that describes the process over time. (Keep 4 places past the decimal in your calculations.)

Solution: There are several approaches to this problem. A linear algebra approach is to solve the system

$$A^T AX = A^T Y \qquad (*)$$

where

$$A = \begin{bmatrix} 0 & 1 \\ 2 & 1 \\ 4 & 1 \\ 6 & 1 \\ 8 & 1 \\ 10 & 1 \end{bmatrix}, \qquad X = \begin{bmatrix} m \\ b \end{bmatrix}, \qquad \text{and} \qquad Y = \begin{bmatrix} 1.00 \\ 0.85 \\ 0.72 \\ 0.59 \\ 0.45 \\ 0.31 \end{bmatrix}.$$

Thus the system $(*)$ is

$$\begin{bmatrix} 0 & 2 & 4 & 6 & 8 & 10 \\ 1 & 1 & 1 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 2 & 1 \\ 4 & 1 \\ 6 & 1 \\ 8 & 1 \\ 10 & 1 \end{bmatrix} \begin{bmatrix} m \\ b \end{bmatrix} = \begin{bmatrix} 0 & 2 & 4 & 6 & 8 & 10 \\ 1 & 1 & 1 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1.00 \\ 0.85 \\ 0.72 \\ 0.59 \\ 0.45 \\ 0.31 \end{bmatrix}$$

or

$$\begin{bmatrix} 220 & 30 \\ 30 & 6 \end{bmatrix} \begin{bmatrix} m \\ b \end{bmatrix} = \begin{bmatrix} 14.82 \\ 3.92 \end{bmatrix}. \qquad (**)$$

Alternatively, we can minimize the square of the errors using partial derivatives. From this we find that we have to solve the equation

$$\begin{bmatrix} \sum_{i=1}^{n} x_i^2 & \sum_{i=1}^{n} x_i \\ \sum_{i=1}^{n} x_i & n \end{bmatrix} \begin{bmatrix} m \\ b \end{bmatrix} = \begin{bmatrix} \sum_{i=1}^{n} x_i y_i \\ \sum_{i=1}^{n} y_i \end{bmatrix}. \qquad (\dagger)$$

To find the values in the system $(\dagger)$, we create a table with some useful computations:

i	x_i	y_i	x_i^2	$x_i y_i$
1	0	1	0	0
2	2	0.85	4	1.7
3	4	0.72	16	2.88
4	6	0.59	36	3.54
5	8	0.45	64	3.6
6	10	0.31	100	3.1
Sum:	30	3.92	220	14.82

Thus the system $(\dagger)$ is

$$\begin{bmatrix} 220 & 30 \\ 30 & 6 \end{bmatrix} \begin{bmatrix} m \\ b \end{bmatrix} = \begin{bmatrix} 14.82 \\ 3.92 \end{bmatrix}, \qquad (\ddagger)$$

which is precisely system $(**)$.

To solve system (∗∗) (or system (‡)) we'll compute the inverse of the square matrix on the left-hand side. That is,

$$\begin{bmatrix} 220 & 30 \\ 30 & 6 \end{bmatrix}^{-1} \begin{bmatrix} 220 & 30 \\ 30 & 6 \end{bmatrix} \begin{bmatrix} m \\ b \end{bmatrix} = \begin{bmatrix} 220 & 30 \\ 30 & 6 \end{bmatrix}^{-1} \begin{bmatrix} 14.82 \\ 3.92 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} m \\ b \end{bmatrix} = \frac{1}{220 \cdot 6 - 30^2} \begin{bmatrix} 6 & -30 \\ -30 & 220 \end{bmatrix} \begin{bmatrix} 14.82 \\ 3.92 \end{bmatrix}$$

$$\begin{bmatrix} m \\ b \end{bmatrix} = \frac{1}{420} \begin{bmatrix} 6(14.82) - 30(3.92) \\ -30(14.82) + 220(3.92) \end{bmatrix}$$

$$= \frac{1}{420} \begin{bmatrix} -28.68 \\ 417.8 \end{bmatrix} \approx \begin{bmatrix} -0.0682857143 \\ 0.9947619048 \end{bmatrix}.$$

Thus the least squares line is roughly $y = -0.06829x + 0.99476$.

(b) Use the model equation you have derived in (a) to predict the time when there is no alcohol remaining.

Solution: There is no alcohol remaining when $y = 0$, so we solve $y = -0.06829x + 0.99476 = 0$. This is straightforward:

$$x \approx -\frac{0.99476}{-0.06829} \approx 14.568.$$

This says that there will be $y = 0$ liters of alcohol left after $x = 14.6$ hours of evaporation.

6 Automobile tires are considered unsafe once the depth of the tread decreases below a certain point. Consider an exponential random variable X that is the safe lifetime (in thousands of miles driven) of a randomly selected tire. Bargain Brand Tires have a particular basic model whose average safe lifetime is 10 thousand miles. (The fact that a car has 4 tires need not be taken into account.)

(a) What is the value of the parameter k for this exponential density?

Solution: We are told that $E(X) = 10,000$. Moreover, we know from our studies of exponential densities that $E(X) = 1/k$. Hence $10,000 = 1/k$, or $k = \frac{1}{10,000} = 10^{-4} = 0.0001$.

(b) Find the probability that a randomly selected tire lasts more than 10,000 miles?

Solution: We are asked for $\Pr(X > 10,000) = 1 - \Pr(X \leq 10,000)$. We may compute this in terms of one of two integrals:

$$\Pr(X > 10,000) = \int_{10,000}^{\infty} ke^{-kx}\,dx \qquad \text{or} \qquad 1 - \Pr(X \leq 10,000) = 1 - \int_{0}^{10,000} ke^{-kx}\,dx.$$

The first integral is improper, but we'll use it anyway as it is more direct:

$$\begin{aligned} \Pr(X > 10,000) &= \int_{10,000}^{\infty} ke^{-kx}\,dx \\ &= \lim_{b\to\infty} \int_{10,000}^{b} ke^{-kx}\,dx \\ &= \lim_{b\to\infty} \left[-e^{-kx}\right]_{10,000}^{b} \\ &= \lim_{b\to\infty} \left[-e^{-kb} - (-e^{-10,000k})\right] \\ &= e^{-1} \qquad \text{since } k = 1/10,000 \\ &\approx 0.36787944. \end{aligned}$$

That is, there is a roughly 36.8% chance that a tire lasts more than $10,000$ miles.

(c) You are going on 10,000 mile adventure around North and Central America. Assume that driving on an unsafe tire is certain to cause a flat. Further assume that having a flat repaired on this particular trip will end up costing \$100.

Your local tire dealer sells Bargain Brand basic model tires for \$75 each (you only need one for this problem). They also offer a \$40 protection plan, so that if you have a flat anywhere on your trip all tire related repairs and towing are covered. Finally, the salesman shows you a deluxe Bargain Brand model for \$110 with an average lifetime of 25 thousand miles and the protection plan thrown in free. **Should you a) buy the basic model and the protection plan, b) buy the basic model without the protection plan or c) buy the deluxe model?** Back up your answer with the appropriate calculation.

Solution: Let us consider the costs associated with our tire purchase. We have three options:

	Costs			
Option	Tire	Plan	Repair	Total
(a)	\$75	\$40	None	\$115
(b)	\$75	No	$\$100 \cdot \Pr(X < 10{,}000)$	B
(c)	\$110	Free!	None	\$110

Clearly option (a) is out – we get a better tire and the protection plan with option (c) for less money. The question is: Is B, the total cost of option (b), more or less than \$110? Well, $B = 75 + 100 \cdot \Pr(X < 10{,}000) = 75 + 100 \cdot (1 - \Pr(X > 10{,}000))$, and we figured out this probability in part (b). We find that $B = 75 + 100(1 - 0.36787944) \approx \$138.21$. Thus option (c) is the best choice.