Final Exam Formula Sheet

Differentiation:

$$\frac{d}{dx}(x^n) = nx^{n-1} (\text{if } n \neq 0)$$

$$\frac{d}{dx}(\ln(|x|)) = \frac{1}{x}$$

$$\frac{d}{dx}(e^{rx}) = re^{rx}$$

$$\frac{d}{dx}(\sin(x)) = \cos(x)$$

$$\frac{d}{dx}(\cos(x)) = -\sin(x)$$

Product Rule: $(uv)' = u'v + uv'$

Quotient Rule: $\left(\frac{u}{v}\right)' = \frac{u'v - uv'}{v^2}$

Chain Rule: $\frac{d}{dx}(f(g(x))) = f'(g(x))g'(x)$

Integration:

$$\int x^n \, dx = \frac{x^{n+1}}{n+1} + C \text{ if } n \neq -1$$

$$\int \frac{1}{x} \, dx = \ln(|x|) + C$$

$$\int e^{rx} \, dx = \frac{1}{r} e^{rx} + C$$

$$\int \cos(x) \, dx = \sin(x) + C$$

$$\int \sin(x) \, dx = -\cos(x) + C$$

Integration by Parts: $\int u \, dv = uv - \int v \, du$

Substitution: $\int f(u(x))u'(x) \, dx = \int f(u) \, du$

Applications of Integrals:

The *average value* of a continuous function $f(x)$ over an interval $[a, b]$ is

$$\text{AV} = \frac{1}{b-a} \int_a^b f(x) \, dx.$$

The *area* A *under the graph* of $y = f(x)$ from a to b is

$$A = \int_a^b f(x) \, dx.$$

Producer's and Consumer's Surplus: If the demand function $p = D(x)$ and the supply function $p = S(x)$ meet at the equilibrium point $E = (x^*, p^*)$, then…

$$CS = \int_0^{x^*} D(x) \, dx - x^* p^* \qquad \text{and} \qquad PS = x^* p^* - \int_0^{x^*} S(x) \, dx.$$

Approximating a Definite Integral:

$$\int_a^b f(x) \, dx \approx f(u_1)\Delta x + f(u_2)\Delta x + \cdots + f(u_n)\Delta x, \qquad \Delta x = \frac{b-a}{n}.$$

Lagrange Multipliers: To maximize (or minimize) a function $z = f(x, y)$ subject to the constraint $g(x, y) = 0$, find the critical points of the function $F(x, y, \lambda) = f(x, y) + \lambda g(x, y)$. That is, solve the system of equations

$$F_x = \frac{\partial F}{\partial x} = 0 \qquad F_y = \frac{\partial F}{\partial y} = 0 \qquad F_\lambda = \frac{\partial F}{\partial \lambda} = g(x, y) = 0$$

for x, y, and λ.

Newton's Law of Cooling:

If $T(t)$ is the temperature of a cooling body at time t in a room with constant air temperature A, then $T'(t) = k(T - A)$.

First Order Linear:

$$y' + P(x)y = G(x) \qquad (SF)$$

has integrating factor $h(x) = e^{\int P(x) \, dx}$.

Second Order Linear:

- $r_1 \neq r_2 \implies y = Pe^{r_1 t} + Qe^{r_2 t}$
- $r_1 = r_2 = r \implies y = e^{rt}(P + Qt)$
- $r = \alpha \pm i\beta \implies y = e^{\alpha t}(P\cos(\beta t) + Q\sin(\beta t))$

Matrices:
The inverse of a square matrix A is the matrix A^{-1} such that $A^{-1}A = AA^{-1} = I$:

$$\text{For a } 2\times 2 \text{ matrix } A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}, \qquad A^{-1} = \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}.$$

In general, elimination (row reduction) may be used to make the following transformation: $[A \mid I] \to [I \mid A^{-1}]$. The three row operations are (1) Interchange any two rows, (2) Multiply any row by a (nonzero) constant, and (3) Add any multiple of one row to another.

Least Squares: The line of best fit $y = mx + b$ to a set of n data points (x_1, y_1), (x_2, y_2), $\ldots$, (x_n, y_n) is found by solving the linear system $A^TAX = A^TY$, where

$$A = \begin{bmatrix} x_1 & 1 \\ x_2 & 1 \\ \vdots & \vdots \\ x_n & 1 \end{bmatrix}, \quad X = \begin{bmatrix} m \\ b \end{bmatrix}, \quad Y = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix} \qquad \text{or} \qquad \begin{bmatrix} \sum_{i=1}^n x_i^2 & \sum_{i=1}^n x_i \\ \sum_{i=1}^n x_i & n \end{bmatrix} \begin{bmatrix} m \\ b \end{bmatrix} = \begin{bmatrix} \sum_{i=1}^n x_i y_i \\ \sum_{i=1}^n y_i \end{bmatrix}.$$

Classifying Critical Points as Local Maxima, Local Minima, or Saddle Points:
Suppose (x_0, y_0) is a critical point of a function $z = f(x,y)$, and $D = f_{xx}f_{yy} - f_{xy}^2$.

- If $D > 0$ at (x_0, y_0) and $f_{xx} < 0$ at (x_0, y_0), then the critical point is a local maximum.
- If $D > 0$ at (x_0, y_0) and $f_{xx} > 0$ at (x_0, y_0), then the critical point is a local minimum.
- If $D < 0$ at (x_0, y_0), then the critical point is a saddle point.

Continuous Random Variables and their density functions:

Distribution	General Continuous	Uniform	Exponential	Normal
Density Function	$f(x)$	$\frac{1}{b-a}$	$\lambda e^{-\lambda x}$	$\frac{1}{\sigma\sqrt{2\pi}}e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}$
Domain	$a \le x \le b$	$a \le x \le b$	$0 \le x < \infty$	$-\infty < x < \infty$
Mean / Variance		$\mu = \frac{a+b}{2}$, $\sigma^2 = \frac{(b-a)^2}{12}$	$\mu = \frac{1}{\lambda}$, $\sigma = \frac{1}{\lambda^2}$	μ and σ

Random Variable X	$\Pr(X)$	$\mu = E(X)$	$\sigma^2 = \text{Var}(X)$
Discrete	$\Pr(X = x_k) = p_k$ Special case: $\Pr(X = x_k) = \frac{1}{N}$	$\sum_{k=1}^N x_k \cdot p_k$ $\frac{x_1 + x_2 + \cdots + x_N}{N}$	 $\frac{(x_1-\mu)^2 + \cdots + (x_N-\mu)^2}{N}$
Continuous	$\Pr(a \le X \le b) = \int_a^b f(x)\,dx$	$\int_a^b xf(x)\,dx$	$\int_a^b (x-\mu)^2 f(x)\,dx$ $= \int_a^b x^2 f(x)\,dx - \mu^2$

Bernoulli Trials: $b(n,k;p) = \binom{n}{k} p^k q^{n-k}$ where $q = 1 - p$.

Chebychev's Theorem: If a distribution X has mean μ and standard deviation σ, then

$$\Pr(\mu - k \le X \le \mu + k) \ge 1 - \frac{\sigma^2}{k^2}.$$