Continuous Probability Functions I

Definition

A *continuous probability density function* $f(x)$ on an interval $[a,b]$ must satisfy two properties:

(1) $f(x) \geq 0$ for all x in $[a,b]$

(2) $\displaystyle\int_a^b f(x)\,dx = 1.$

These two properties can be reformulated as

(1) Probabilities cannot be negative – they must be zero or positive, and

(2) The total probability of a sample space is 1.

(Here a can be $-\infty$ and b can be $+\infty$.)

One typical kind of problem gives a function $f(x)$ and an interval $[a,b]$ and asks whether or not this function satisfies (1) and (2) (that is, whether or not $f(x)$ is a continuous probability density function). Here are some of these sorts of problems:

1 One of the following three functions is a continuous probability density function on the interval $[0,5]$. Figure out which one, and explain why the others are not.

(a) $f(x) = \dfrac{3}{20}(x^2 - 6x + 8)$ (b) $f(x) = \dfrac{1}{6}(1+x)$ (c) $f(x) = \dfrac{2}{5} - \dfrac{2}{25}x$

2 For each of the following functions $f(x)$ and intervals $[a,b]$, find k so that $f(x)$ is a continuous probability density function on $[a,b]$.

(a) $f(x) = k$ on $[0,5]$

(b) $f(x) = \begin{cases} \frac{1}{2}x & \text{if } 0 \leq x \leq 1 \\ \frac{1}{2} - kx & \text{if } 1 < x \leq 3 \end{cases}$ on $[0,3]$

(c) $f(x) = ke^{-3x}$ on $[0,\infty)$

(d) $f(x) = \dfrac{k}{x^2}$ on $[1,4]$

Calculating Probabilities

Given a random variable X with continuous probability density function $f(x)$ on an interval $[a, b]$, the probability that X is between c and d is

$$\Pr(c \leq X \leq d) = \int_c^d f(x)\, dx.$$

(Here we need $a \leq c \leq d \leq b$.)

3 **(The uniform distribution and the uniform density function)** Let $f(x)$ be the probability density function

$$f(x) = \begin{cases} \frac{1}{b-a} & \text{if } a \leq x \leq b \\ 0 & \text{if } x < a \text{ or } x > b. \end{cases}$$

Then $f(x)$ is called the *uniform density function* and the resulting random variable X is said to have a *uniform distribution*.

(a) Calculate $\Pr(0 \leq X \leq 2)$ if X is uniformly distributed with $a = 0$ and $b = 10$.

(b) Calculate $\Pr(0 \leq X \leq \pi)$ if X is uniformly distributed with $a = 0$ and $b = 10$.

(c) Find d so that $\Pr(0 \leq X \leq d) = 0.3$, if X is once again uniformly distributed with $a = 0$ and $b = 10$.

(d) Repeat (a), (b), and (c) with $a = -3$ and $b = 6$.

4 **(The exponential distribution and the exponential density function)** Let $f(x)$ be the probability density function

$$f(x) = \begin{cases} \lambda e^{-\lambda x} & \text{if } x \geq 0 \\ 0 & \text{if } x < 0. \end{cases}$$

(Here λ, the Greek letter lambda, is a parameter that will have more meaning for us next time.) Then $f(x)$ is called the *exponential density function* and the resulting random variable X is said to have an *exponential distribution*. Suppose X is exponentially distributed with parameter $\lambda = 3$.

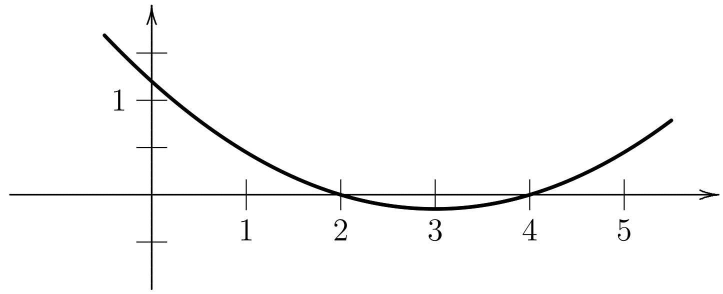
(a) Find the probability $\Pr(0 \leq X \leq 1)$. (Notice that this is the same as $\Pr(X \leq 1)$.)

(b) Find the probability $\Pr(X \geq 1)$ in two ways. First, by direct computation; and second, by using part (a).

(c) Repeat (a) and (b) using $\lambda = 5$.

(d) Something to think about: what is $\Pr(X = 1)$?

1 (a) This is not a probability density function on $[0,5]$ since $f(x) < 0$ for x in the interval $(2,4)$. Here's a quick graph of the function:



(b) This $f(x)$ is not a probability density function because it doesn't integrate to 1:

$$\int_0^5 \frac{1}{6}(1+x)\,dx = \frac{1}{6}\cdot\frac{1}{2}(1+x)^2\Big|_0^5 = \frac{1}{12}\Big[(1+5)^2-(1+0)^2\Big] = \frac{35}{12}.$$

(c) This $f(x)$ is a probability density function on $[0,5]$.

2 (a) $k=\frac{1}{5}$ (b) $k=\frac{1}{16}$ (c) $k=3$ (d) $k=\frac{4}{3}$

3 (a) $\Pr(0\le X\le 2)=\frac{2}{10}$ (b) $\Pr(0\le X\le \pi)=\frac{\pi}{10}$ (c) $d=3$

Part(d):

(a) $\Pr(0\le X\le 2)=\frac{2}{9}$ (b) $\Pr(0\le X\le \pi)=\frac{\pi}{9}$ (c) $d=2.7$

4 (a) $\Pr(0\le X\le 1)=1-e^{-3}\approx 0.950213$ (b) $\Pr(X\ge 1)=e^{-3}\approx 0.049787$

Part(c):

(a) $\Pr(0\le X\le 1)=1-e^{-5}\approx 0.993262$ (b) $\Pr(X\ge 1)=e^{-5}\approx 0.006738$

(d) $\Pr(X=1)=\int_1^1 f(x)\,dx = 0$