Second Order Linear DEs

Second Order Linear Homogeneous Equations (with constant coefficients)

1. Write the equation in standard form:

$$ay'' + by' + cy = 0$$

2. Form the *characteristic equation*

$$ar^2 + br + c = 0. \qquad (*)$$

3. Find the two roots r_1 and r_2 of $(*)$:

$$r_1, r_2 = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

Solutions (here P and Q are constants):

Case 1: $r_1 \neq r_2$ but both are real:

$$y = Pe^{r_1 t} + Qe^{r_2 t}$$

Case 2: $r_1 = r_2 = r$ (so both are real):

$$y = e^{rt}(P + Qt)$$

Case 3: $r_1, r_2 = \alpha \pm i\beta$:

$$y = e^{\alpha t}(P\cos(\beta t) + Q\sin(\beta t))$$

1 Verify that $y = 2e^{-2t} + 3e^{5t}$ is a solution of $y'' - 3y' - 10y = 0$.

2 Verify that $y = te^{3t}$ is a solution of $y'' - 6y' + 9y = 0$.

3 Verify that $y = 2e^{-3t}\cos(4t)$ is a solution of $y'' + 6y' + 25y = 0$.

For each of the following differential equations:

(a) Find the general solution, then

(b) Solve the given initial value problem.

4 $y'' - 2y' - 3y = 0,$
$y(0) = 4, \qquad y'(0) = 0$

5 $y'' = 4y,$
$y(0) = 3, \qquad y'(0) = 2$

6 $y'' - 2y' + y = 0,$
$y(0) = 5, \qquad y'(0) = 3$

7 $y'' + 2y' + 5y = 0,$
$y(0) = 2, \qquad y'(0) = 0$

8 $y'' + 5y' = 0,$
$y(0) = 1, \qquad y'(0) = -1$

9 $y'' = -9y,$
$y(0) = 6, \qquad y'(0) = 3$

Answers:

4 (a) $y = Pe^{3t} + Qe^{-t}$ (b) $y = e^{3t} + 3e^{-t}$ **5** (a) $y = Pe^{2t} + Qe^{-2t}$ (b) $y = 2e^{2t} + e^{-2t}$

6 (a) $y = e^t(P + Qt)$ (b) $y = e^t(5 - 2t)$ **7** (a) $y = e^{-t}(P\cos(2t) + Q\sin(2t))$

(b) $y = e^{-t}(2\cos(2t) + \sin(2t))$ **8** (a) $y = P + Qe^{-5t}$ (b) $y = \frac{4}{5} + \frac{1}{5}e^{-5t}$

9 (a) $y = P\cos(3t) + Q\sin(3t)$ (b) $y = 6\cos(3t) + \sin(3t)$