Applications of Differential Equations

Newton's Law Of Cooling

Newton's Law of Cooling says that the rate a body cools (or heats up) is proportional to the difference between the body's temperature and the ambient temperature.

1. Suppose I bring my piping hot (200° F) morning cup of tea out into the 40° F Cleveland weather. I'd like to drink the tea when it cools to 150° F. How long do I have to wait for my tea?

   (a) Write down a differential equation involving the temperature T of the tea, the time t (in minutes), and the ambient temperature A.

   (b) Solve this differential equation.

   (c) What is the initial condition? This will allow us to solve for *one* of the constants in our equation.

   (d) We should still have a constant in the solution of the IVP above. Suppose the tea is 175° after 5 minutes. When will the tea be 150° F?

Interest

2. My retirement account pays 4% annual interest, compounded continuously. I (and my employer) make deposits of $5,000 per year. Suppose I opened the account with an initial deposit of $20,000. Assume deposits are made continuously.

   Let $A(t)$ be the amount in the account at time t years since the initial deposit.

   (a) Set up a differential equation satisfied by $A(t)$. What is the initial condition? That is, write down the initial value problem.

   (b) Solve for $A(t)$.

Mixture Problems

3. Suppose we have a 200 gallon tank in which there is 120 gallons of water. There are 12 pounds of salt dissolved in the water. Pure water is being added at a rate of 2 gallons per minute and the (well-stirred) mixture is being poured out at 1 gallon per minute.

   (a) Let $Q(t)$ be the quantity of salt (in pounds) in the vat at time t, and let $C(t)$ be the concentration of salt (lbs/gal) at time t. How do $C(t)$ and $Q(t)$ relate? (In this problem, you might need an expression for $V(t)$, the volume of liquid in the vat at time t.)

   (b) Write down a differential equation involving $Q(t)$ and its derivative. What is the initial condition?

   (c) Solve for $Q(t)$, and find an expression for $C(t)$.

   (d) We're willing to accept that the water is "pure" if has less than 0.0002 pounds of salt per gallon. (This is close the EPA guideline of 20 mg/L.) How long will it take for our vat to contain only pure water? Will the vat have overflowed?

Newton's Law Of Cooling

1 (a) $T'(t) = k\,(T(t) - A)$ or $T'(t) = k\,(T(t) - 40)$ or $\dfrac{dT}{dt} = k\,(T - 40)$

(b) $T(t) = 40 + Ce^{kt}$

(c) $T(0) = 200$, so $C = 160$. That is, $T(t) = 40 + 160e^{kt}$.

(d) $k = \frac{1}{5}\ln\left({}^{135}\!/_{160}\right) = \frac{1}{5}\ln\left({}^{27}\!/_{32}\right) \approx -0.03397981$

$T = 150$ when $t = \frac{\ln\left({}^{110}\!/_{160}\right)}{k} = \frac{\ln\left({}^{11}\!/_{16}\right)}{\frac{1}{5}\ln\left({}^{27}\!/_{32}\right)} \approx 11.027$ minutes.

Interest

2 (a) $A'(t) = 5000 + 0.04A(t)$ or $\frac{dA}{dt} = 5000 + 0.04A$ or $A' - 0.04A = 5000$
$A(0) = 20,000$.

(b) $A(t) = -125,000 + 145,000e^{0.04t}$ or $A(t) = 20,000e^{0.04t} + 125,000\,(e^{0.04t} - 1)$

Mixture Problems

3 (a) $C(t) = \frac{Q(t)}{V(t)}$ where $V(t) = 120 + t$. Thus $C(t) = \frac{Q(t)}{120+t}$.

(b) $Q'(t) = -C(t)$ or $Q'(t) = -\frac{Q(t)}{120+t}$. $Q(0) = 12$.

(c) $Q(t) = \frac{1440}{120+t}$ and so $C(t) = \frac{1440}{(120+t)^2}$

(d) $C(t) = 0.0002$ when $(120+t)^2 = \frac{1440}{0.0002} = 7,200,000$, or when $t = \sqrt{\frac{1440}{0.0002}} - 120 \approx 2563.28$ minutes. The vat will be full in 80 minutes (when $V(t) = 200$ gallons), and by that time the water will still be too salty to be "pure."

Another approach is to find the concentration at $t = 80$ minutes, when the vat reaches capacity. At this time, $C(80) = \frac{1440}{(120+80)^2} = 0.036$ pounds of salt per gallon. This is still 180 times the "purity" limit.